19 (b) $\quad V=\mathbb{R}^3,\ u=(2,1,3) \quad W=\{(x,y,z): x+3y-2z=0\}.$

First find an o.n.b. for $W=\{(x,y,z): x=-3y+2z\}=\text{Span}\left\{\underbrace{\begin{pmatrix}-3\\1\\0\end{pmatrix}}_{w_1},\underbrace{\begin{pmatrix}2\\0\\1\end{pmatrix}}_{w_2}\right\}$

Use Gram-Schmidt: $\quad v_1=w_1$

$$v_2=w_2-\frac{\langle w_2,v_1\rangle}{\|v_1\|^2}v_1=\begin{pmatrix}2\\0\\1\end{pmatrix}-\frac{-6}{10}\begin{pmatrix}-3\\1\\0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}10-9\\0+3\\5-0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}1\\3\\5\end{pmatrix}.$$

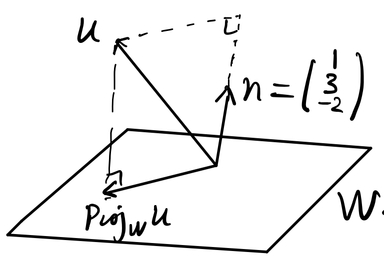
$$\Rightarrow u_1=\frac{v_1}{\|v_1\|}=\frac{1}{\sqrt{10}}\begin{pmatrix}-3\\1\\0\end{pmatrix},\quad u_2=\frac{v_2}{\|v_2\|}=\frac{1}{\sqrt{1^2+3^2+5^2}}\begin{pmatrix}1\\3\\5\end{pmatrix}=\frac{1}{\sqrt{35}}\begin{pmatrix}1\\3\\5\end{pmatrix}$$ for an o.n.b. for W

$$\text{Proj}_W u=\langle u,u_1\rangle u_1+\langle u,u_2\rangle u_2=\frac{\langle u,v_1\rangle}{\|v_1\|^2}v_1+\underbrace{\frac{\langle u,v_2\rangle}{\|v_2\|^2}v_2}_{=\frac{\langle u,5v_2\rangle}{\|5v_2\|^2}\cdot 5v_2}$$

$$=\frac{-6+1+0}{10}\begin{pmatrix}-3\\1\\0\end{pmatrix}+\frac{2\cdot1+1\cdot3+3\cdot5}{1^2+3^2+5^2}\begin{pmatrix}1\\3\\5\end{pmatrix}$$

$$=-\frac{1}{2}\begin{pmatrix}-3\\1\\0\end{pmatrix}+\frac{20}{35}\begin{pmatrix}1\\3\\5\end{pmatrix}=\begin{pmatrix}\frac{4}{7}+\frac{3}{2}\\\frac{12}{7}-\frac{1}{2}\\\frac{20}{7}-0\end{pmatrix}=\begin{pmatrix}\frac{29}{14}\\\frac{17}{14}\\\frac{20}{7}\end{pmatrix}=\frac{1}{14}\begin{pmatrix}29\\17\\40\end{pmatrix}.$$

Alternatively



$W=n^\perp=\{v\in\mathbb{R}^3,\ \langle v,n\rangle=0\}.$

$$\text{Proj}_W u=u-\text{Proj}_{\text{Span}\{n\}}u=u-\frac{\langle u,n\rangle}{\|n\|^2}n$$

$$=\begin{pmatrix}2\\1\\3\end{pmatrix}-\frac{2\cdot1+1\cdot3+3\cdot(-2)}{1^2+3^2+(-2)^2}\begin{pmatrix}1\\3\\-2\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\frac{1}{14}\begin{pmatrix}1\\3\\-2\end{pmatrix}=\frac{1}{14}\begin{pmatrix}29\\17\\40\end{pmatrix}$$

✓

(c). $V = P(\mathbb{R})$. $h(x) = 4+3x-2x^2$, $W = P_1(\mathbb{R})$.

First use Gram-Schmidt to find o.n.b. for $W = P_1(\mathbb{R}) = \mathrm{Span}\{\underset{w_1}{1}, \underset{w_2}{x}\}$.

$$v_1 = w_1 = 1. \qquad \|v_1\|^2 = \int_0^1 1\cdot 1\,dx = 1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 \qquad \langle w_2, v_1\rangle = \int_0^1 x\cdot 1\,dx = \frac{1}{2}$$

$$v_2 = x - \frac{\frac{1}{2}}{1}\cdot 1 = x - \frac{1}{2}.$$

$\Rightarrow \{v_1, v_2\}$ is an orthogonal basis for $W \Leftrightarrow \left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}\right\}$ is an o.n.b. for W

$$\Rightarrow \mathrm{Proj}_W h = \frac{\langle h, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle h, v_2\rangle}{\|v_2\|^2} v_2$$

$$\langle h, v_1\rangle = \int_0^1 (4+3x-2x^2)\cdot 1\,dx = 4 + \frac{3}{2} - \frac{2}{3} = \frac{1}{6}(24+9-4) = \frac{29}{6}.$$

$$\langle h, v_2\rangle = \int_0^1 (4+3x-2x^2)(x-\tfrac{1}{2})\,dx = \int_0^1 \left(-2 + x\cdot(4-\tfrac{3}{2}) + x^2\cdot(3+1) + x^3\cdot(-2)\right) dx$$

$$= -2 + \frac{5}{2}\cdot\frac{1}{2} + \frac{4}{3} - \frac{2}{4} = \frac{1}{12}(-24+15+16-6) = \frac{1}{12}$$

$$\|v_2\|^2 = \int_0^1 (x-\tfrac{1}{2})^2 dx = \int_0^1 (x^2 - x + \tfrac{1}{4})\,dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{4-6+3}{12} = \frac{1}{12}$$

$$\Rightarrow \mathrm{Proj}_W h = \frac{\frac{29}{6}}{1}\cdot 1 + \frac{\frac{1}{12}}{\frac{1}{12}}\cdot(x - \tfrac{1}{2}) = x + \frac{29-3}{6} = x + \frac{13}{3}.$$

21. orthonormal basis for $P_2(\mathbb{R})$ from Example 5:

$$u_1 = \frac{1}{\sqrt{2}}, \quad u_2 = \sqrt{\frac{3}{2}}\,x, \quad u_3 = \sqrt{\frac{5}{8}}(3x^2-1).$$

$$\text{Proj}_W h = \langle h, u_1\rangle u_1 + \langle h, u_2\rangle u_2 + \langle h, u_3\rangle u_3$$ ("best" 2nd degree approximation of h)

$$\langle h, u_1\rangle = \int_{-1}^{1} e^t \cdot \frac{1}{\sqrt{2}}\,dt = \frac{1}{\sqrt{2}} e^t\Big|_{-1}^{1} = \frac{1}{\sqrt{2}}(e-e^{-1}).$$

$$\langle h, u_2\rangle = \int_{-1}^{1} e^t \cdot \sqrt{\frac{3}{2}}\,t\,dt = \sqrt{\frac{3}{2}}\int_{-1}^{1} t\,de^t = \sqrt{\frac{3}{2}}\left(t\cdot e^t\Big|_{-1}^{1} - \int_{-1}^{1} e^t dt\right)$$

$$= \sqrt{\frac{3}{2}}\cdot\left(e^1 - (-e^{-1}) - \left(e^t\Big|_{-1}^{1}\right)\right) = \sqrt{\frac{3}{2}}\left(e+e^{-1} - (e-e^{-1})\right) = \sqrt{\frac{3}{2}}\cdot 2e^{-1}$$

$$\langle h, u_3\rangle = \int_{-1}^{1} e^t \cdot \sqrt{\frac{5}{8}}(3t^2-1)\,dt = \sqrt{\frac{5}{8}}\left(\int_{-1}^{1} 3\cdot t^2 de^t - \left(e^t\Big|_{-1}^{1}\right)\right)$$

$$= \sqrt{\frac{5}{8}}\cdot\left(-(e-e^{-1}) + 3\cdot\left(t^2e^t\Big|_{-1}^{1} - \int_{-1}^{1} e^t\cdot 2t\,dt\right)\right)$$

$$= \sqrt{\frac{5}{8}}\left(-e+e^{-1} + 3\cdot(e-e^{-1}) - 6\int_{-1}^{1} t\,de^t\right)$$

$$= \sqrt{\frac{5}{8}}\left(2e - 2e^{-1} - 6\cdot\left(te^t\Big|_{-1}^{1} - \int_{-1}^{1} e^t dt\right)\right)$$

$$= \sqrt{\frac{5}{8}}\cdot 2\cdot\left(e - e^{-1} - 3\left(e-(-e^{-1})\right) + 3\cdot(e-e^{-1})\right)$$

$$= \sqrt{\frac{5}{8}}\cdot 2\cdot(e-7e^{-1}).$$

$$\Rightarrow \text{Proj}_W h = \frac{1}{\sqrt{2}}(e-e^{-1})\cdot\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}}\cdot 2e^{-1}\cdot\sqrt{\frac{3}{2}}\,x + \sqrt{\frac{5}{8}}\cdot 2(e-7e^{-1})\sqrt{\frac{5}{8}}(3x^2-1)$$

$$= \frac{1}{2}(e-e^{-1}) + 3e^{-1}x + \frac{5}{4}(e-7e^{-1})(3x^2-1)$$

$$= \left(\frac{1}{2}-\frac{5}{4}\right)e + \left(-\frac{1}{2}+\frac{35}{4}\right)e^{-1} + 3e^{-1}x + \frac{15}{4}(e-7e^{-1})x^2$$

$$= \left(-\frac{3}{4}e + \frac{33}{4}e^{-1}\right) + 3e^{-1}x + \frac{15}{4}(e-7e^{-1})x^2.$$

6.3 2(a) $V=\mathbb{R}^3$, $g(a_1,a_2,a_3)=a_1-2a_2+4a_3$

$$= \langle (a_1,a_2,a_3),(1,-2,4)\rangle$$

$\Rightarrow$ $y=(1,-2,4)$.

2(c) $V=P_2(\mathbb{R})$, $\langle f,h\rangle=\int_0^1 f(t)h(t)\,dt$, $g(f)=f(0)+f'(1)$.

$\underset{\langle y,f\rangle}{\overset{}{g(f)=\langle f,y\rangle}}$, Assume $\{u_1,u_2,u_3\}$ is an o.n.b. for $P_2(\mathbb{R})$.

$$y=\langle y,u_1\rangle u_1+\langle y,u_2\rangle u_2+\langle y,u_3\rangle u_3$$

$$=g(u_1)u_1+g(u_2)u_2+g(u_3)u_3.$$

From 6.2.2(c), we found an o.n.b.:

$$u_1=1,\quad u_2=\sqrt{3}(2x-1),\quad u_3=\sqrt{5}(6x^2-6x+1).$$

$$\Rightarrow \underset{u_1(0)+u_1'(1)}{g(u_1)}=1+0=1,\quad \underset{u_2(0)+u_2'(0)}{g(u_2)}=\underset{\sqrt{3}}{-\sqrt{3}+\sqrt{3}\cdot 2},\quad g(u_3)=\sqrt{5}\cdot\left(1+(12x-6)\big|_{x=1}\right)$$

$$=\sqrt{5}\cdot 7$$

$$\Rightarrow y=1\cdot 1+\sqrt{3}\cdot\sqrt{3}(2x-1)+\sqrt{5}\cdot 7\cdot\sqrt{5}(6x^2-6x+1)$$

$$=1+(6x-3)+35(6x^2-6x+1)=33-6\cdot 34x+35\cdot 6x^2$$

$$=33-204\cdot x+210\cdot x^2.$$

From (the notes of) class, we have another o.n.b.:

$\tilde{u}_1 = \sqrt{5}\cdot x^2$, $\tilde{u}_2 = \sqrt{3}(4x - 5x^2)$, $\tilde{u}_3 = 3 - 12x + 10x^2$.

$\Rightarrow g(\tilde{u}_1) = 0 + \sqrt{5}\cdot 2 = 2\sqrt{5}$, $g(\tilde{u}_2) = \sqrt{3}\cdot(4 - 10x|_{x=1}) = -6\sqrt{3}$

$g(\tilde{u}_3) = 3 + (-12 + 20x|_{x=1}) = 11$.

$\Rightarrow y = 2\sqrt{5}\cdot\sqrt{5}x^2 - 6\sqrt{3}\cdot\sqrt{3}\cdot(4x - 5x^2) + 11\cdot(3 - 12x + 10x^2)$

$= 10x^2 - 72x + 90x^2 + 33 - 132x + 110x^2$

$= 33 - 204x + 210x^2$ same as before ✓.

3(a): $V = \mathbb{R}^2$, $T(a,b) = (2a+b, a-3b)$, $x = (3,5)$

$T^*x = \langle T^*x, e_1\rangle e_1 + \langle T^*x, e_2\rangle e_2$ $\quad Te_1 = T(1,0) = (2,1)$

$= \langle x, Te_1\rangle e_1 + \langle x, Te_2\rangle e_2$ $\quad Te_2 = T(0,1) = (1,-3)$

$= (3\cdot 2 + 5\cdot 1)(1,0) + (3\cdot 1 + 5\cdot(-3))\cdot(0,1)$

$= (11, -12)$.

3(c). $V = P_1(\mathbb{R})$, $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$, $T(f) = f' + 3f$.

$f(t) = 4-2t$.

For any orthonormal basis $\{u_1, u_2\}$ of $P_1(\mathbb{R})$,

$$T^*f = \langle T^*f, u_1\rangle u_1 + \langle T^*f, u_2\rangle u_2$$

$$= \langle f, Tu_1\rangle u_1 + \langle f, Tu_2\rangle u_2.$$

Use Gram-Schmidt to $\{\underset{w_1}{\underset{\parallel}{1}}, \underset{w_2}{\underset{\parallel}{t}}\}$ to get $\{u_1, u_2\}$:

$v_1 = w_1 = 1$, $\quad \|v_1\|^2 = \int_{-1}^{1} 1\cdot 1\,dt = t\big|_{-1}^{1} = 2$

$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 \quad \langle w_2, v_1\rangle = \int_{-1}^{1} t\cdot 1\,dt = 0$

$v_2 = w_2 - \frac{0}{\|v_1\|^2} v_1 = w_2 = t$, $\|v_2\|^2 = \int_{-1}^{1} t\cdot t\,dt = \frac{1}{3}t^3\big|_{-1}^{1} = \frac{2}{3}$

$\Rightarrow \quad u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}, \quad u_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{3}{2}}\cdot t.$

$\Rightarrow Tu_1 = u_1' + 3u_1 = 0 + 3\cdot\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

$Tu_2 = u_2' + 3u_2 = \sqrt{\frac{3}{2}} + 3\cdot\sqrt{\frac{3}{2}}\, t = \sqrt{\frac{3}{2}}(1+3t)$

$\langle f, Tu_1\rangle = \int_{-1}^{1} (4-2t)\cdot\frac{3}{\sqrt{2}}\,dt = \frac{3}{\sqrt{2}}\cdot 8$

$$\langle f, Tu_2\rangle = \int_{-1}^{1} (4-2t)\cdot\sqrt{\tfrac{3}{2}}\cdot(1+3t)\,dt$$

$$(8-(2-(-2)))$$

$$= \sqrt{\tfrac{3}{2}}\cdot\int_{-1}^{1} (4+10t-6t^2)\,dt = \sqrt{\tfrac{3}{2}}\cdot\Big(8-\overset{\overset{(8-(2-(-2)))}{\|}}{\Big(2t^3\Big|_{-1}^{1}\Big)}\Big)$$

$$= \sqrt{\tfrac{3}{2}}\cdot 4$$

$$\Rightarrow T^*f = \frac{24}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + \sqrt{\tfrac{3}{2}}\cdot 4\cdot\sqrt{\tfrac{3}{2}}\,t = 12+6t.$$

Check: $\forall\ g(t) = c_1 + c_2 t \in P_1(\mathbb{R})$. $Tg(t) = g' + 3g = c_2 + 3(c_1+c_2 t) = (3c_1+c_2) + 3c_2\cdot t$

$$\langle T^*f, g\rangle = \int_{-1}^{1} (12+6t)\cdot(c_1+c_2 t)\,dt = \int_{-1}^{1}(12c_1 + *\cdot t + 6c_2 t^2)\,dt$$

$$= 24c_1 + 6c_2\Big(\frac{t^3}{3}\Big|_{-1}^{1}\Big) = 24c_1 + 4c_2.$$

$$\langle f, Tg\rangle = \langle 4-2t, (3c_1+c_2)+3c_2 t\rangle = \int_{-1}^{1}(4-2t)\cdot\big((3c_1+c_2)+3c_2\cdot t\big)\,dt$$

$$= \int_{-1}^{1}\big(4(3c_1+c_2) + *\cdot t - 6c_2\cdot t^2\big)\,dt$$

$$= 8\cdot(3c_1+c_2) - 6c_2\cdot\frac{t^3}{3}\Big|_{-1}^{1} = 24c_1 + (8-4)c_2 = 24c_1 + 4c_2 \quad ✓$$

20(a) $A = \begin{pmatrix} 1 & -3 \\ 1 & -2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$

$$A^T \cdot A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -3 \\ 1 & -2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -4 & 14 \end{pmatrix} = 2 \begin{pmatrix} 2 & -2 \\ -2 & 7 \end{pmatrix}$$

$$\Rightarrow (A^T A)^{-1} = \frac{1}{2} \cdot \frac{1}{2 \cdot 7 - 4} \begin{pmatrix} 7 & 2 \\ 2 & 2 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 7 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^T y = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ -38 \end{pmatrix} = 2 \cdot \begin{pmatrix} 9 \\ -19 \end{pmatrix}$$

$$\Rightarrow (A^T A)^{-1} A^T y = \frac{1}{20} \cdot 2 \cdot \begin{pmatrix} 7 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 9 \\ -19 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 25 \\ -20 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -2 \end{pmatrix} = v_1 = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

$\Rightarrow$ best-fit linear function $y = c_0 + c_1 x = \frac{5}{2} - 2 \cdot x$.

Error: $\|Av_1 - y\|^2 = \langle Av_1 - y, Av_1 - y \rangle$

$$= \langle Av_1, Av_1 - y \rangle - \langle y, Av_1 \rangle + \|y\|^2$$

$$= 0 - \langle A^T y, v_1 \rangle + \|y\|^2$$

$$E = \|Av_1 - y\|^2 = \|y\|^2 - \langle A^T y, v_1 \rangle$$

$$\|y\|^2 = 9^2 + 6^2 + 2^2 + 1^2 = 81 + 36 + 5 = 122$$

$$\langle A^T y, v_1 \rangle = 2 \cdot \left(9 \cdot \frac{5}{2} + 19 \cdot 2\right) = 45 + 76 = 121$$

$$\Rightarrow E = 122 - 121 = 1.$$

Alternatively, find an o.n.b. for $R(A) = \text{Span}\left\{ \underbrace{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}_{= w_1}, \underbrace{\begin{pmatrix}-3\\-2\\0\\1\end{pmatrix}}_{= w_2} \right\}$

different with v_1 above $\Rightarrow$ $v_1 = w_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ $\quad \|v_1\|^2 = 4$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = \begin{pmatrix}-3\\-2\\0\\1\end{pmatrix} - \frac{-4}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}.$$

$\|v_2\|^2 = 4+1+1+4 = 10.$ $\quad y = \begin{pmatrix}9\\6\\2\\1\end{pmatrix}$, $\langle y, v_2\rangle = -18-6+2+2 = -20$

$$\underbrace{\text{Proj}_{R(A)}\, y}_{= y_1} = \frac{\langle y, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle y, v_2\rangle}{\|v_2\|^2} v_2 = \frac{18}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \underbrace{\frac{-20}{10}\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}}_{= \begin{pmatrix}4\\2\\-2\\-4\end{pmatrix}}$$

$$= \frac{1}{2}\begin{pmatrix}17\\13\\5\\1\end{pmatrix}.$$

Solve $A\begin{pmatrix}c_0\\c_1\end{pmatrix} = y_1 = \frac{1}{2}\begin{pmatrix}17\\13\\5\\1\end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & -3 & 17 \\ 1 & -2 & 13 \\ 1 & 0 & 5 \\ 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & -3 & 12 \\ 0 & -2 & 8 \\ 0 & 1 & -4 \end{array}\right) \Rightarrow \begin{pmatrix}b\\k\end{pmatrix} = \begin{pmatrix}5\\-4\end{pmatrix}\cdot\frac{1}{2} = \begin{pmatrix}\frac{5}{2}\\-2\end{pmatrix}.$$

Error: $\|y - y_1\|^2 = \left\| \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix} \right\|^2 = 1.$ ✓.

$\underline{\text{Ex}}$: $\{(t_i, y_i)\} = \{(-2,4), (-1,3), (0,1), (1,-1), (2,-3)\}$.

$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$, $y = \begin{pmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -3 \end{pmatrix}$. $A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}$.

$A^Ty = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -18 \end{pmatrix}$. $(A^TA)^{-1}A^Ty = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 4 \\ -18 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 \\ -9 \end{pmatrix} = \underbrace{\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}}_{= x_1}$.

best-fit linear function: $y = \frac{4}{5} - \frac{9}{5}t$.

$\Rightarrow y_1 = Ax_1 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} 4 \\ -9 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 22 \\ 13 \\ 4 \\ -5 \\ -14 \end{pmatrix}$.

Error $= \|y - y_1\|^2 = \left\| \begin{pmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -3 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 22 \\ 13 \\ 4 \\ -5 \\ -14 \end{pmatrix} \right\|^2 = \frac{1}{25} \cdot \left\| \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\|^2 = \frac{4+4+1+1}{25} = \frac{2}{5}$.

To find y_1, we can also project to $W = R(A) = \text{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{= w_1}, \underbrace{\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{= w_2} \right\}$

$v_1 = w_1$. $\|v_1\|^2 = 5$

$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \frac{0}{5} * = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$. $y = \begin{pmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -3 \end{pmatrix}$

$y_1 = \langle y, u_1 \rangle u_1 + \langle y, u_2 \rangle u_2 = \frac{\langle y, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle y, v_2 \rangle}{\|v_2\|^2} v_2$

$= \frac{4}{5}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{-8-3-1-6}{4+1+1+4} \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \frac{4}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{9}{5}\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 22 \\ 13 \\ 4 \\ -5 \\ -14 \end{pmatrix}$ ✓

6.5 2(d) $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$.

Find eigenvalues and eigenspaces:

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = \begin{vmatrix} 0 & \lambda+2 & -\lambda\cdot\frac{\lambda}{2}+2 \\ 0 & -\lambda-2 & 2+\lambda \\ 2 & 2 & -\lambda \end{vmatrix} = 2\cdot(\lambda+2)\begin{vmatrix} \lambda+2 & \frac{1}{2}(-\lambda^2+4) \\ -1 & 1 \end{vmatrix}$$

$$=2\cdot(\lambda+2)\cdot(\lambda+2)\cdot\begin{vmatrix} 1 & \frac{1}{2}(2-\lambda) \\ -1 & 1 \end{vmatrix} = 2(\lambda+2)^2\cdot\left(1+\frac{1}{2}(2-\lambda)\right)$$

$$= (\lambda+2)^2\cdot(4-\lambda) = 0 \quad \Rightarrow \quad \begin{matrix} \lambda=-2, & m_{-2}=2 \\ \lambda=4, & m_4=1 \end{matrix}$$

$\lambda=-2$: $A-(-2)I = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_{-2} = \mathrm{Span}\left\{ \overset{w_1}{\overset{\parallel}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}}, \overset{w_2}{\overset{\parallel}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}} \right\}$.

$\lambda=4$: $A-4I = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_4 = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Find o.n.b. for E_{-2}: $v_1 = w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\|v_1\|^2 = 2$

$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$.

$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$.

o.n.b. for E_4: $u_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$.

$$\Rightarrow P = (u_1\ u_2\ u_3) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \text{ satisfies } P^tAP = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

2(e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 0 & (\lambda-2)+1 & (\lambda-2)(2-\lambda)+1 \\ 0 & 1-\lambda & -1+\lambda \\ 1 & 1 & 2-\lambda \end{vmatrix} = 1\cdot(\lambda-1)\cdot\begin{vmatrix} \lambda-1 & -\lambda^2+4\lambda-3 \\ -1 & 1 \end{vmatrix}$$

$$= (\lambda-1)\cdot\begin{vmatrix} \lambda-1 & -(\lambda-1)(\lambda-3) \\ -1 & 1 \end{vmatrix} = (\lambda-1)^2\cdot\begin{vmatrix} 1 & -\lambda+3 \\ -1 & 1 \end{vmatrix} = (\lambda-1)^2\cdot(1-\lambda+3) = 0.$$

$\Rightarrow \lambda = 1 \quad m_1 = 2.$ and $\lambda = 4$, $m_4 = 1$.

$$\lambda=1:\ A-1\cdot I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_1 = \text{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

$$\lambda=4:\ A-4I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_4 = \text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

Eigenspaces are the same as that of 2(d) $\Rightarrow$ same o.n.b.

$$\Rightarrow P = (u_1\ u_2\ u_3) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \text{ satisfies } P^tAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$