4(d): $A = \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix}$

- Find eigenvalues:

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -3 & 1 & 2 \\ -2 & 1-\lambda & -1 & 2 \\ -2 & 1 & -1-\lambda & 2 \\ -2 & -3 & 1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 0 & (1-\lambda)\cdot(-\frac{\lambda}{2})-3 & \frac{\lambda}{2}+1 & -\lambda+2 \\ -2 & 1-\lambda & -1 & 2 \\ 0 & \lambda & -\lambda & 0 \\ 0 & \lambda-4 & 2 & 2-\lambda \end{vmatrix}$$

$$= -(-2)\cdot \begin{vmatrix} \frac{1}{2}\lambda^2-\frac{\lambda}{2}-3 & \frac{1}{2}(\lambda+2) & -(\lambda-2) \\ \lambda & -\lambda & 0 \\ \lambda-4 & 2 & 2-\lambda \end{vmatrix} = 2\cdot\lambda\cdot(\lambda-2) \begin{vmatrix} \frac{1}{2}(\lambda^2-\lambda-6) & \frac{1}{2}(\lambda+2) & -1 \\ 1 & -1 & 0 \\ \lambda-4 & 2 & -1 \end{vmatrix}$$

$$= 2\lambda\cdot(\lambda-2)\cdot \begin{vmatrix} \overset{\frac{1}{2}(\lambda+2)(\lambda-2)}{\frac{1}{2}(\lambda^2-4)} & \frac{1}{2}(\lambda+2) & -1 \\ 0 & -1 & 0 \\ \lambda-2 & 2 & -1 \end{vmatrix} = 2\lambda\cdot(\lambda-2)\cdot(-1)\cdot(\lambda-2)\cdot \begin{vmatrix} \frac{1}{2}(\lambda+2) & -1 \\ 1 & -1 \end{vmatrix}$$

$$= -2\lambda\cdot(\lambda-2)^2\cdot(\underbrace{-\tfrac{1}{2}(\lambda+2)+1}_{\shortparallel \atop -\lambda}) = \lambda^2\cdot(\lambda-2)^2 = 0.$$

$$\Rightarrow \quad \lambda = 0, \quad m_0 = 2; \quad \lambda = 2, \quad m_2 = 2$$

- Find Jordan blocks for each eigenvalue:

$$\lambda = 0: \quad A - 0I \longrightarrow \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & -1 & 1 & -2 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow N(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \dim E_0 = 1 < m_0 = 2.$$

$\Rightarrow$ dot diagram for $\lambda = 0$: $\begin{matrix} v_1 & \bullet \\ \uparrow & \\ v_2 & \bullet \end{matrix}$ $\Leftrightarrow$ 1 Jordan block: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Find $v_2 \in N((A-0I)^2) - N(A-0I)$:

$$(A-0I)^2 = \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -8 & 4 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & -8 & 4 & 8 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 2 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow N(A^2) = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$\rightsquigarrow$ choose $v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

$$\Rightarrow v_1 = A \cdot v_2 = 1 \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \\ -3 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \in N(A - 0 \cdot I).$$

- $\lambda = 2$:

$$A - 2I = \begin{pmatrix} -2 & -3 & 1 & 2 \\ -2 & -1 & -1 & 2 \\ -2 & 1 & -3 & 2 \\ -2 & -3 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 3 & -1 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 2 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow N(A - 2I) = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{v_3}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_4} \right\}$$

$\Rightarrow \dim E_2 = 2 = m_2 \Rightarrow$ dot diagram $\underset{v_3}{\bullet} \quad \underset{v_4}{\bullet}$

$\Leftrightarrow$ 2 Jordan blocks of size 1

- The Jordan canonical form of A is

$$J = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & & \\ & \boxed{2} & \\ & & \boxed{2} \end{pmatrix}. \qquad Q = (v_1\ v_2\ v_3\ v_4) = \begin{pmatrix} -1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

satisfies $Q^{-1} \cdot A \cdot Q = J$.

5(c): $T: P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$, $T(f(x)) = f''(x) + 2f(x)$.

- Choose standard basis $\beta = \{1, x, x^2, x^3\}$.

$$[T] = \big([T(1)]_\beta \quad [T(x)]_\beta \quad [T(x^2)]_\beta \quad [T(x^3)]_\beta \big)$$

$$\begin{matrix} \| & \| & \| & \| \\ [2]_\beta & [2x]_\beta & [2+2x^2]_\beta & [6\cdot x + 2x^3]_\beta \end{matrix}$$

$$= \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = A.$$

- $\det(A - \lambda I) = (2-\lambda)^4 = 0 \Rightarrow \lambda = 2, \quad m_2 = 4.$

- $A - 2I = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow N(A - 2I) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\quad \dim E_2 = 2 < 4 = m_2$

- $(A - 2I)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = O_{4\times 4}$

$\Rightarrow N((A-2I)^2) = \mathbb{R}^4$, $\dim N((A-2I)^2) = 4 = 2 + \underset{\substack{\uparrow \\ \#\text{ of dots on 2nd row}}}{2}$

$\Rightarrow$ dot diagram

$v_1 \bullet \quad \bullet\, v_3$

$v_2 \bullet \quad \bullet\, v_4$

choose $v_2, v_4 \in N((A-2I)^2) - N(A-2I)$:

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow v_1 = (A-2I)v_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = (A-2I)v_4 = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix}$$

- Jordan canonical form of T: $J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} \end{pmatrix}$

  Jordan canonical basis:

$$\{ \underset{\substack{\Updownarrow \\ v_1}}{2}, \underset{\substack{\Updownarrow \\ v_2}}{x^2}, \underset{\substack{\Updownarrow \\ v_3}}{6x}, \underset{\substack{\Updownarrow \\ v_4}}{x^3} \}$$

verify: $Tv_1 = T(2) = 0 + 2 \cdot 2 = 2v_1$

$Tv_2 = T(x^2) = 2 + 2x^2 = v_1 + 2v_2$

$Tv_3 = T(6x) = 0 + 2 \cdot 6x = 2v_3$

$Tv_4 = T(x^3) = 6x + 2 \cdot x^3 = 6v_3 + 2v_4$ ✓

6. . $\det(A-\lambda I) = \det(A^t - \lambda I)$

$\Rightarrow$ A and A^t have the same characteristic polynomial.

$\Rightarrow$ A and A^t have the same eigenvalues with same multiplicities

. For any eigenvalue λ, and any positive integer r

$$\operatorname{rank}((A-\lambda I)^r) = \operatorname{rank}(((A-\lambda I)^r)^t) = \operatorname{rank}((A^t-\lambda I)^r)$$

$$\Rightarrow \quad \operatorname{nullity}((A-\lambda I)^r) = \operatorname{nullity}((A^t-\lambda I)^r).$$

Assuming the characteristic polynomial splits

$\Rightarrow$ A and A^t have the same dot diagram

$\Rightarrow$ A and A^t have the same Jordan canonical form J

$\Rightarrow$ $A \sim J \sim A^t$: A and A^t are similar ∎

6.1: 3. $f(t)=t,\ g(t)=e^t \in C([0,1])$

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt = \int_0^1 t\cdot e^t dt = \int_0^1 t\, de^t$$

$$= t\cdot e^t\Big|_0^1 - \int_0^1 e^t dt = e - e^t\Big|_0^1 = e-(e-1) = 1.$$

$$\|f\|^2 = \langle f,f\rangle = \int_0^1 t\cdot t\,dt = \frac{1}{3}t^3\Big|_0^1 = \frac{1}{3} \Rightarrow \|f\| = \frac{1}{\sqrt{3}}$$

$$\|g\|^2 = \langle g,g\rangle = \int_0^1 e^t\cdot e^t dt = \frac{1}{2}e^{2t}\Big|_0^1 = \frac{1}{2}(e^2-1) \Rightarrow \|g\| = \sqrt{\frac{e^2-1}{2}}$$

$$\|f+g\|^2 = \langle f+g, f+g\rangle = \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle$$

$$= \|f\|^2 + 2\langle f,g\rangle + \|g\|^2 = \frac{1}{3} + 2\cdot 1 + \frac{e^2-1}{2} = \frac{11}{6} + \frac{e^2}{2}$$

$$\Rightarrow \|f+g\| = \left(\frac{11}{6}+\frac{e^2}{2}\right)^{\frac{1}{2}}.$$

Cauchy-Schwarz: $|\langle f,g\rangle|^2 = 1$. $\|f\|^2\cdot\|g\|^2 = \frac{1}{3}\cdot\frac{e^2-1}{2} = \frac{e^2-1}{6}$

$$e^2-1 > 2.7^2-1 = 7.29-1 = 6.29 > 6 \Rightarrow |\langle f,g\rangle|^2 \le \|f\|^2\|g\|^2$$

Triangle inequality: $\frac{1}{\sqrt{3}} \sim 0.577$ $\sqrt{\frac{e^2-1}{2}} \sim 1.787$

$$\left(\frac{11}{6}+\frac{e^2}{2}\right)^{\frac{1}{2}} \sim 2.351 < \underbrace{0.577+1.787}_{2.364}$$

$$\Rightarrow \quad \|f+g\| < \|f\| + \|g\|$$

9. (a) Assume $\langle x, z\rangle = 0 \quad \forall z \in \beta = \{v_1, v_2, \dots, v_n\}$

β is a basis $\Rightarrow \quad x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$$\Rightarrow \underset{\|x\|^2}{\overset{\shortparallel}{\langle x, x\rangle}} = \langle x, a_1 v_1 + a_2 v_2 + \cdots + a_n v_n\rangle$$

$$= a_1\langle x, v_1\rangle + a_2\langle x, v_2\rangle + \cdots + a_n\langle x, v_n\rangle$$

$$= a_1 \cdot 0 + a_2 \cdot 0 + \cdots + a_n \cdot 0 = 0$$

$\Rightarrow \quad x = 0.$

(b) $\langle x, z\rangle = \langle y, z\rangle \quad \forall z \in \beta$

$\Rightarrow \langle x-y, z\rangle = \langle x, z\rangle - \langle y, z\rangle = 0, \quad \forall z \in \beta$

$\overset{(a)}{\Rightarrow} \quad x - y = 0 \quad$ i.e. $x = y$.

11. $\|x+y\|^2 + \|x-y\|^2$

$= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$

$= 2\langle x, x\rangle + 2\langle y, y\rangle = 2\|x\|^2 + 2\|y\|^2.$ (Parallelogram law)

20 (a) $\frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$

$= \frac{1}{4}\left(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - \left(\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle\right)\right)$

$= \frac{1}{4}\left(2\langle x, y\rangle + 2\langle y, x\rangle\right) = \langle x, y\rangle$ (polar identity)

6.2(a) $S = \{ \underset{w_1}{\underset{\|}{(1,0,1)}}, \underset{w_2}{\underset{\|}{(0,1,1)}}, \underset{w_3}{\underset{\|}{(1,3,3)}} \}$.

$v_1 = w_1 = (1,0,1)$

$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = (0,1,1) - \frac{1}{2}(1,0,1) = (-\frac{1}{2}, 1, \frac{1}{2}) = \frac{1}{2}(-1,2,1)$

$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 = \frac{\langle w_3, 2v_2 \rangle}{\|2v_2\|^2}(2v_2)$

$= (1,3,3) - \frac{4}{2}(1,0,1) - \frac{8}{6}(-1,2,1)$

$= (\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}) = \frac{1}{3}(1,1,-1)$

$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1,0,1)$

$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}(-1,2,1)$

$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{3}}(1,1,-1)$

$\beta = \{u_1, u_2, u_3\}$ is an orthonormal basis for $\text{span}(S) = \mathbb{R}^3$.

Fourier coefficients of $x = (1,1,2)$:

$\langle x, u_1 \rangle = \frac{1}{\sqrt{2}} \cdot 3$, $\langle x, u_2 \rangle = \frac{1}{\sqrt{6}} \cdot 3$, $\langle x, u_3 \rangle = \frac{1}{\sqrt{3}} \cdot 0 = 0$

$\Rightarrow x = \frac{3}{\sqrt{2}} u_1 + \frac{3}{\sqrt{6}} u_2 + 0 \cdot u_3$

check: $\frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,0,1) + \frac{3}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}(-1,2,1) = \frac{3}{2}(1,0,1) + \underset{\frac{1}{2}}{\frac{3}{6}}(-1,2,1) = (1,1,2)$ ✓.

2(c) $V = P_2(\mathbb{R})$. $S = \{\underset{w_1}{\underset{\|}{1}}, \underset{w_2}{\underset{\|}{x}}, \underset{w_3}{\underset{\|}{x^2}}\}$, $h(x) = 1 + x$.

$v_1 = w_1 = 1$, $\qquad \|v_1\|^2 = \int_0^1 1\cdot 1\,dx = x\Big|_0^1 = 1$

$v_2 = w_2 - \dfrac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1$ $\qquad \langle w_2, v_1\rangle = \int_0^1 x\cdot 1\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

$$v_2 = x - \frac{\frac{1}{2}}{1}\cdot 1 = x - \frac{1}{2}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$\langle w_3, v_1\rangle = \int_0^1 x^2\cdot 1\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$.

$\langle w_3, v_2\rangle = \int_0^1 x^2(x - \frac{1}{2})\,dx = \frac{1}{4}x^4 - \frac{1}{6}x^3\Big|_0^1 = \frac{1}{12}$.

$\|v_2\|^2 = \int_0^1 (x - \frac{1}{2})\cdot(x - \frac{1}{2}) = \int_0^1 (x^2 - x + \frac{1}{4})dx = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right)\Big|_0^1 = \frac{4-6+3}{12} = \frac{1}{12}$

$$v_3 = x^2 - \frac{\frac{1}{3}}{1}\cdot 1 - \frac{\frac{1}{12}}{\frac{1}{12}}\cdot(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

$\|v_3\|^2 = \int_0^1 (x^2 - x + \frac{1}{6})^2 dx = \int_0^1 \left(x^4 + x^2 + \frac{1}{36} - 2x^3 + \frac{x^2}{3} - \frac{x}{3}\right)dx$

$= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} + \frac{1}{9} - \frac{1}{6} = \frac{1}{180}(36 + 60 + 5 - 90 + 20 - 30) = \frac{1}{180}$

$\Rightarrow u_1 = \dfrac{v_1}{\|v_1\|} = 1$, $\quad u_2 = \dfrac{v_2}{\|v_2\|} = \dfrac{x - \frac{1}{2}}{\frac{1}{\sqrt{12}}} = 2\sqrt{3}\cdot(x - \frac{1}{2}) = 2\sqrt{3}x - \sqrt{3}$.

$u_3 = \dfrac{v_3}{\|v_3\|} = \dfrac{x^2 - x + \frac{1}{6}}{\frac{1}{6\sqrt{5}}} = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$.

$\beta = \{u_1, u_2, u_3\}$ is an orthonormal basis for $P_2(\mathbb{R})$.

Fourier coefficients for h relative to β:

$$a_i = \langle h, u_i \rangle = \frac{\langle h, v_i \rangle}{\|v_i\|}, \quad h = a_1 u_1 + a_2 u_2 + a_3 u_3$$

$$\langle h, v_1 \rangle = \int_0^1 (1+x) \cdot 1 \, dx = \left(x + \tfrac{1}{2}x^2\right)\Big|_0^1 = \frac{3}{2}$$

$$\langle h, v_2 \rangle = \int_0^1 (1+x) \cdot \left(-\tfrac{1}{2} + x\right) dx = \int_0^1 \left(x^2 + \tfrac{1}{2}x - \tfrac{1}{2}\right) dx = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}.$$

$$\langle h, v_3 \rangle = \int_0^1 (1+x) \cdot \left(\tfrac{1}{6} - x + x^2\right) dx = \int_0^1 \left(\tfrac{1}{6} - \tfrac{5}{6}x + \underbrace{x^2 \cdot (1-1)}_{=0} + x^3 \cdot 1\right) dx$$

$$= \left(\tfrac{1}{6}x - \tfrac{5}{12}x^2 + \tfrac{1}{4}x^4\right)\Big|_0^1 = \frac{1}{12}(2 - 5 + 3) = 0.$$

$$\Rightarrow a_1 = \frac{\frac{3}{2}}{1} = \frac{3}{2}, \quad a_2 = \frac{\frac{1}{12}}{\frac{1}{\sqrt{12}}} = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}, \quad a_3 = \frac{0}{\|v_3\|} = 0.$$

$$\Rightarrow \underline{h = \frac{3}{2} \cdot u_1 + \frac{\sqrt{3}}{6} \cdot u_2 + 0 \cdot u_3}$$

Check: $\frac{3}{2} \cdot 1 + \frac{\sqrt{3}}{6} \cdot (2\sqrt{3}x - \sqrt{3}) = x + 1 = h(x)$ ✓.