7.1 3(a): $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = 2f(x) - f'(x)$.

$\beta = \{1, x, x^2\}$

$$A = [T]_\beta = \left(\underset{\substack{\| \\ [2]_\beta}}{[T(1)]_\beta} \quad \underset{\substack{\| \\ [2x-1]_\beta}}{[T(x)]_\beta} \quad \underset{\substack{\| \\ [2x^2-2x]_\beta}}{[T(x^2)]_\beta} \right) = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}.$$

· char. polynomial of A: $f(t) = (2-t)^2 \Rightarrow$ eigenvalue $\lambda = 2$

multiplicity $\text{mult}(2) = 3$.

· Calculate the eigenspace $E_2 = N(A-2I)$:

$$A-2I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N(A-2I) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$\Rightarrow$ Dot diagram:

0

$\uparrow$

$\bullet\ (A-2I)^2 v_1 = v_3$

$\uparrow$

$\bullet\ (A-2I)v_1 = v_2$

$\uparrow$

$\bullet\ v_1$

Find $v \notin N((A-2I)^2)$

$$(A-2I)^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N((A-2I)^2) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\Rightarrow$ choose $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $\Rightarrow f_1(x) = x^2$

$$\Rightarrow (A-2I) \cdot v_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = v_2 \Rightarrow f_2(x) = -2x = (T-2I)(f_1(x))$$

$$\Rightarrow (A-2I)^2 v_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = v_3 \Rightarrow f_3(x) = 2 = (T-2I)^2(f_1(x))$$

$\Rightarrow$ cycle of generalized eigenvectors:

$$\left\{ (T-2I)^2(f_1(x)),\ (T-2I)(f_1(x)),\ f_1(x) \right\}$$

$$\parallel$$

$$\left\{ 2,\ -2x,\ x^2 \right\}.$$

Jordan form: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$

<u>check</u>: $T(2) = 2\cdot 2.$

$T(-2x) = 2\cdot(-2x) - (-2) = 2 + 2\cdot(-2x).$

$T(x^2) = 2\cdot x^2 - 2x = 1\cdot(-2x) + 2\cdot x^2$ ✓.

7.1.4. Let $\gamma = \{ (T-\lambda I)^{p-1}(v), (T-\lambda I)^{p-2}(v), \cdots, (T-\lambda I)(v), v \}$.

be a cycle of generalized eigenvectors of T corresponding to the eigenvalue λ (Definition on pg. 482 of the text book)

In particular $(T-\lambda I)^p(v) = 0$.

Prove: $\text{Span}(\gamma)$ is a T-invariant subspace of V.

<u>Proof</u>: It is enough to prove $T(w) \in \text{Span}(\gamma)$, for any $w \in \gamma$.

Set $w_k = (T-\lambda I)^k v$, for some $k = 0, 1, \cdots, p-1$.

$$
\begin{aligned}
T(T-\lambda I)^k(v) &= ((T-\lambda I) + \lambda I) \cdot (T-\lambda I)^k v \\
&= (T-\lambda I)^{k+1} v + \lambda \cdot (T-\lambda I)^k v.
\end{aligned}
$$

So $T w_k = w_{k+1} + \lambda \cdot w_k$ if $k = 0, 1, \cdots, p-2$.

If $k = p-1$, then $T(w_{p-1}) = \lambda \cdot w_{p-1}$ because

$(T-\lambda I)^p v = (T-\lambda I)((T-\lambda I)^{p-1} v) = 0$ by the definition of (assumption) the cycle of generalized eigenvectors.

∎

7.2. 2 :

$\lambda_1 = 2$ $\qquad$ $\lambda_2 = 4$ $\qquad$ $\lambda_3 = -3$

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array} \qquad \begin{array}{cc} \bullet & \bullet \\ \bullet & \\ \bullet & \end{array} \qquad \begin{array}{cc} \bullet & \bullet \end{array}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} \; \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} \; \boxed{2} \qquad \boxed{\begin{matrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{matrix}} \; \boxed{4} \qquad \boxed{-3} \; \boxed{-3}$$

$$\Rightarrow J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & & & & & & \\ & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & & & & & \\ & & \boxed{2} & & & & \\ & & & \boxed{\begin{matrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{matrix}} & & & \\ & & & & \boxed{4} & & \\ & & & & & \boxed{-3} & \\ & & & & & & \boxed{-3} \end{pmatrix}$$

3. $$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 2 & 1 \\ \cdot & 2 \end{matrix}} & & \\ & & \boxed{3} & \\ & & & \boxed{3} \end{pmatrix}$$

(a) characteristic polynomial $= (2-t)^5 \cdot (3-t)^2$

(b) dot diagram: $\lambda_1 = 2$ $\qquad$ $\lambda_2 = 3$

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array} \qquad\qquad \begin{array}{cc} \bullet & \bullet \end{array}$$

(c) $\lambda_1 = 2$: $\dim E_2 = 2$, $\dim K_2 = 5 \Rightarrow E_2 \neq K_2$

$\lambda_2 = 3$: $\dim E_3 = 2 = \dim K_3 \Longrightarrow E_3 = K_3$.

(d): $\lambda_1 = 2$: $K_2 = N((T-2I)^3) \supsetneq N((T-2I)^2)$. $\quad P_1 = 3$

$\lambda_2 = 3$: $K_3 = N(T-3I) = E_3$. $\quad P_2 = 1$.

(e) $U_i = (T - \lambda_i I)|_{K_{\lambda_i}}$

$\lambda_1 = 2$: $\text{nullity}(U_1) = \dim((T-\lambda_1 I)|_{K_{\lambda_1}}) = 2$

$\text{nullity}(U_1^2) = \dim((T-\lambda_1 I)^2|_{K_{\lambda_1}}) = 2+2 = 4$.

$\text{rank}(U_1) = \dim K_{\lambda_1} - \text{nullity}(U_1) = 5-2 = 3$

$\text{rank}(U_1^2) = \dim K_{\lambda_1} - \text{nullity}(U_1^2) = 5-4 = 1$.

$\lambda_2=3$: $\quad \text{nullity}(U_2)=2$

$\text{nullity}(U_2^2)=\dim((T-\lambda_2 I)^2(K_{\lambda_2}))=\dim(K_{\lambda_2})=2$

$\text{rank}(U_2)=2-2=0$

$\text{rank}(U_2^2)=2-2=0.$

4(a) $\quad A=\begin{pmatrix}-3 & 3 & -2\\ -7 & 6 & -3\\ 1 & -1 & 2\end{pmatrix}$

$-1\cdot(3+t)+3 \qquad (2-t)\cdot(3+t)-2 = -t^2-t+4$

· $f(t)=\begin{vmatrix}-3-t & 3 & -2\\ -7 & 6-t & -3\\ 1 & -1 & 2-t\end{vmatrix}=\begin{vmatrix}0 & -t & -t^2-t+4\\ 0 & -1-t & 11-7t\\ 1 & -1 & 2-t\end{vmatrix}$

$=1\cdot\begin{vmatrix}-t & -(t^2+t-4)\\ -(t+1) & 11-7t\end{vmatrix}=-1\cdot\left(t\cdot(11-7t)+(t^2+t-4)\cdot(t+1)\right)$

$=-1\cdot(11t-7t^2+t^3+t^2+t^2+t-4t-4)$

$=-(t^3-5t^2+8t-4)=-(t^3-t^2-(4t^2-8t+4))$

$=-t^2(t-1)+4(t-1)^2=-(t-1)\cdot(t^2-4t+4)=-(t-1)\cdot(t-2)^2$

$\Rightarrow \lambda=1,\ 2.$

· $\lambda=1$: $\quad A-I=\begin{pmatrix}-4 & 3 & -2\\ -7 & 5 & -3\\ 1 & -1 & 1\end{pmatrix}\to\begin{pmatrix}0 & -1 & 2\\ 0 & -2 & 4\\ 1 & -1 & 1\end{pmatrix}\to\begin{pmatrix}1 & -1 & 1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{pmatrix}$

$$\to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \quad K_1 = E_1 = \operatorname{Span}\left\{ \overset{=u}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \right\}$$

$$\cdot\ \lambda = 2: \quad A-2I = \begin{pmatrix} -5 & 3 & -2 \\ -7 & 4 & -3 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -2 & -2 \\ 0 & -3 & -3 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_2 = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$\Rightarrow \quad \begin{array}{l} \bullet\ v_2 \\ \uparrow {\scriptstyle A-2I} \\ \bullet\ v_1 \end{array} \qquad \text{choose } v_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \text{ and solve}$$

$$(A-2I)v_1 = v_2: \quad \left(\begin{array}{ccc|c} -5 & 3 & -2 & -1 \\ -7 & 4 & -3 & -1 \\ 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & -2 & -2 & 4 \\ 0 & -3 & -3 & 6 \\ 1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{pick } v_1 = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} u & v_2 & v_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ 1 & 1 & 0 \end{pmatrix} \text{ satisfies } Q^{-1}AQ = \begin{pmatrix} \boxed{1} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} \end{pmatrix}$$

<u>OR</u>: we can pick any $\underset{\neq 0}{v_1} \in N((A-2I)^2) - N(A-2I)$.

and calculate $v_2 = (A-2I)v_1$.

5.(c) $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ $\gamma = \{1, x, x^2, x^3\}$.

$T(f(x)) = f''(x) + 2f(x)$.

$$[T]_\beta = \left(\underset{\substack{\| \\ [2]_\gamma}}{[T(1)]_\gamma} \quad \underset{\substack{\| \\ [2x]_\gamma}}{[T(x)]_\gamma} \quad \underset{\substack{\| \\ [2+2x^2]_\gamma}}{[T(x^2)]_\gamma} \quad \underset{\substack{\| \\ (6x+2x^3)_\gamma}}{[T(x^3)]_\gamma} \right)$$

$$= \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = A$$

- $f(t) = \det(A - tI) = (2-t)^4 \Rightarrow \lambda = 2$.
- $A - 2I = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow N(A-2I) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. $\dim N(A-2I) = 2$.

$$(A-2I)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0 \text{ matrix}$$

$$\Rightarrow \dim N((A-2I)^2) = \dim N(L_0) = \dim \mathbb{R}^4 = 4.$$

$\Rightarrow$ dot diagram:
$$\begin{array}{cc} v_3 = (A-2I)v_1 & (A-2I)v_2 = v_4 \\ \bullet & \bullet \\ \bullet & \bullet \\ v_1 & v_2 \end{array}$$
$\Rightarrow$ Jordan canonical Form
$$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} \end{pmatrix}$$

- Choose $v_1, v_2 \in N((A-2I)^2) - N(A-2I)$ that are linearly independent (modulo $N(A-2I)$):

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow (A-2I)v_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = v_3$$

(v_3 and v_4 are linearly independ.)

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow (A-2I)v_2 = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix} = v_4$$

$$\Rightarrow Q = (v_3 \ v_1 \ v_4 \ v_2) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ satisfies}$$

$$Q \cdot A \cdot Q^{-1} = J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} \end{pmatrix}.$$

$\Rightarrow$ Jordan canonical basis: $\{2, x^2, 6x, x^3\}$.

Check: $T(2) = 2\cdot 2$, $T(x^2) = 2 + 2\cdot x^2 = 1\cdot 2 + 2\cdot x^2$ ✓

$T(6x) = 2\cdot 6x$, $T(x^3) = 6x + 2\cdot x^3 = 1\cdot(6x) + 2\cdot x^3$. ∎

10. characteristic polynomial of T splits $\Rightarrow$ T has a Jordan canonical form.

Let λ be an eigenvalue of T.

(a) Prove: $\dim(K_\lambda) =$ sum of the lengths of all the blocks corresponding to λ in the Jordan canonical form of T.

(b) Deduce that $E_\lambda = K_\lambda \Leftrightarrow$ all Jordan blocks corresponding to λ are 1×1 matrices.

Proof: (a) Let $\beta = \{v_1, \cdots, v_d, u_1, \cdots, u_k\}$ be a Jordan canonical basis for T such that $v_1, \cdots, v_d$ are vectors associated to Jordan blocks corresponding to λ.

Then $\{v_1, \cdots, v_d\}$ is a basis for K_λ, which is a disjoint union of cycles of generalized eigenvectors corresponding to the eigenvalue λ.

Each such a cycle has a length equal to the size of the corresponding Jordan block.

$\Rightarrow$ $\dim K_\lambda = d =$ sum of lengths of all the blocks corresponding to λ.

(b) $E_\lambda = K_\lambda \Leftrightarrow$ each cycle has length 1

$\Leftrightarrow$ all Jordan blocks corresponding to λ are 1×1 matrices.