5.2.7: $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Check whether A is diagonalizable.

- $f(t) = \det(A - tI) = \begin{vmatrix} 1-t & 4 \\ 2 & 3-t \end{vmatrix} = (t-1)(t-3) - 8 = t^2 - 4t - 5 = (t-5)(t+1)$

$\Rightarrow$ eigenvalues $\lambda = -1, 5$.

- Find eigenvectors:

$\lambda = -1$: $A - 1 \cdot I = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda = 5$: $A - 5 \cdot I = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow Q = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$ satisfies $Q^{-1} A \cdot Q = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = D$

$\Rightarrow A = Q \cdot D \cdot Q^{-1}$. $\quad Q^{-1} = \frac{1}{-3} \cdot \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$

$$\Rightarrow A^n = Q \cdot D^n \cdot Q^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} \cdot \frac{1}{3} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{n+1} \cdot 2 & 5^n \\ (-1)^n & 5^n \end{pmatrix} \cdot \frac{1}{3} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \cdot \begin{pmatrix} (-1)^n \cdot 2 + 5^n & (-1)^{n+1} \cdot 2 + 2 \cdot 5^n \\ (-1)^{n+1} + 5^n & (-1)^n + 2 \cdot 5^n \end{pmatrix}$$

check: $n=2$: $A^2 = \frac{1}{3} \cdot \begin{pmatrix} 2+25 & -2+50 \\ -1+25 & 1+50 \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}$

$\parallel$

$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}$ ✓

9.(a). $[T]_\beta$ is upper triangular:

$$A=[T]_\beta=\begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ & \ddots & \\ O & & a_{nn} \end{pmatrix}$$

$$\Rightarrow \det(A-tI)=\begin{vmatrix} a_{11}-t & & * \\ & a_{22}-t & \\ & & \ddots \\ O & & a_{nn}-t \end{vmatrix}=(a_{11}-t)\cdot(a_{22}-t)\cdots(a_{nn}-t)$$

$\Rightarrow$ characteristic polynomial of $T=\det(A-tI)$ splits.

13. (a) λ is an eigenvalue of $T \Leftrightarrow \exists v\neq 0$ s.t. $Tv=\lambda v$

(T invertible $\Rightarrow N(T)=\{0\}\Rightarrow \lambda\neq 0$) $\Leftrightarrow \exists v\neq 0$ s.t. $T^{-1}v=\lambda^{-1}v$

$\Leftrightarrow \lambda$ is an eigenvalue of T^{-1}.

- eigenspace of T for $\lambda = N(T-\lambda I)$

Because $T-\lambda I=\lambda\cdot\lambda^{-1}\cdot T-\lambda\cdot T\cdot T^{-1}=-\lambda T\cdot(T^{-1}-\lambda^{-1}I)$ and T is invertible, $\lambda\neq 0$

$N(T-\lambda I)=N(T^{-1}-\lambda^{-1}I)=$ eigenspace of T^{-1} for λ^{-1}.

(b) T is diagonalizable $\Leftrightarrow \exists$ a basis $\beta=\{v_1,v_2,\cdots,v_n\}$ consisting of eigenvectors for T

$\Rightarrow \beta=\{v_1,v_2,\cdots,v_n\}$ consists of eigenvectors for T^{-1}

$\Rightarrow T^{-1}$ is diagonalizable.

OR: T is diagonalizable $\Leftrightarrow [T]_\gamma = A$ is diagonalizable for a chosen basis γ

$\Leftrightarrow \exists Q$ s.t. $Q^{-1}AQ = D$ diagonal matrix

$\Leftrightarrow \exists Q$ s.t. $Q^{-1} \cdot A^{-1} \cdot Q = D^{-1}$ diagonal

$\Leftrightarrow [T^{-1}]_\gamma = [T]_\gamma^{-1}$ is diagonalizable

$\Leftrightarrow$ T is diagonalizable.

5.4 6(d): $V = M_{2\times 2}(\mathbb{R})$, $T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$. $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

T-cyclic subspace generated by $z = \text{Span}\{z, Tz, T^2z, \cdots\} = W$

$z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Tz = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$$T^2z = T \cdot Tz = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \cdot Tz$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 3 \cdot Tz$$

$\Rightarrow T^k z = 3^{k-1} \cdot Tz$ for $k \geq 1$. (by Induction)

$\Rightarrow W = \text{Span}\{z, Tz\} = \text{Span}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\right\}$.

13: <u>Proof</u>: $w \in W = \text{Span}\{v, Tv, T^2v, \cdots\}$

$\Longrightarrow \exists\, k_1, \cdots, k_r \in \{0, 1, 2, \cdots\}$ and $C_i \in F, i = 1, \cdots, r$.

s.t. $w = C_1 \cdot T^{k_1} v + C_2 \cdot T^{k_2} v + \cdots + C_r \cdot T^{k_r} v$

$= (C_1 \cdot T^{k_1} + C_2 \cdot T^{k_2} + \cdots + C_r T^{k_r})\, v$

$= g(T)(v)$

where $g(t) = c_1 \cdot t^{k_1} + c_2 \cdot t^{k_2} + \cdots + c_r t^{k_r}$ is a polynomial.

Conversely. if $g(t) = a_0 + a_1 \cdot t + \cdots + a_r \cdot t^r$ satisfies

$w = g(T)(v) = a_0 v + a_1 T v + \cdots + a_r T^r v$

then clearly $w \in \text{Span}\{v, Tv, \cdots, T^r v\} \subset W$. ∎

18: $\det(A-tI) = f(t) = (-1)^n t^n + a_{n-1} \cdot t^{n-1} + \cdots + a_1 t + a_0$

(a) $f(0) = a_0 = \det(A)$

A is invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow a_0 \neq 0$.

(b) By Cayley-Hamilton Theorem,

$$f(A) = (-1)^n \cdot A^n + a_{n-1} \cdot A^{n-1} + \cdots + a_1 \cdot A + a_0 \cdot I_n = 0$$

$$= A \cdot ((-1)^n \cdot A^{n-1} + a_{n-1} \cdot A^{n-2} + \cdots + a_1 \cdot I_n) + a_0 \cdot I_n = 0$$

$$\Rightarrow (-a_0)^{-1} A \cdot ((-1)^n \cdot A^{n-1} + a_{n-1} \cdot A^{n-2} + \cdots + a_1 I_n) = I_n$$

$$\Rightarrow A^{-1} = -\frac{1}{a_0} ((-1)^n \cdot A^{n-1} + a_{n-1} \cdot A^{n-2} + \cdots + a_1 \cdot I_n) \quad \blacksquare$$

(c) $A=\begin{pmatrix}1&2&1\\0&2&3\\0&0&-1\end{pmatrix}$ $n=3$, $A^2=\begin{pmatrix}1&2&1\\0&2&3\\0&0&-1\end{pmatrix}\begin{pmatrix}1&2&1\\0&2&3\\0&0&-1\end{pmatrix}=\begin{pmatrix}1&6&6\\0&4&3\\0&0&1\end{pmatrix}$.

$$f(t)=\begin{vmatrix}1-t&2&1\\0&2-t&3\\0&0&-1-t\end{vmatrix}=(1-t)\cdot(2-t)\cdot(-1-t)=-(t^2-3t+2)\cdot(t+1)$$
$$=-(t^3-3t^2+2t+t^2-3t+2)$$
$$=-t^3+2t^2+t-2.$$

$\Rightarrow a_0=-2,\ a_1=1,\ a_2=2.$

$$\Rightarrow A^{-1}=-\frac{1}{a_0}\cdot\left((-1)^3\cdot A^2+a_2\cdot A+a_1\cdot I_3\right)$$
$$=-\frac{1}{-2}\cdot\left(-\begin{pmatrix}1&6&6\\0&4&3\\0&0&1\end{pmatrix}+2\cdot\begin{pmatrix}1&2&1\\0&2&3\\0&0&-1\end{pmatrix}+1\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$$
$$=\frac{1}{2}\cdot\begin{pmatrix}2&-2&-4\\0&1&3\\0&0&-2\end{pmatrix}$$

Check: $\begin{pmatrix}1&2&1\\0&2&3\\0&0&-1\end{pmatrix}\cdot\frac{1}{2}\cdot\begin{pmatrix}2&-2&-4\\0&1&3\\0&0&-2\end{pmatrix}$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$ ✓.