5.1.5(e) $\quad T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$$T(f(x)) = x\cdot f'(x) + f(2)\cdot x + f(3)$$

Let $\gamma = \{1, x, x^2\}$ be the standard basis.

First calculate $[T]_\gamma = \left([T(1)]_\gamma \quad [T(x)]_\gamma \quad [T(x^2)]_\gamma \right)$

$$[T(1)]_\gamma: \; x\cdot 0 + 1\cdot x + 1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad [T(x)]_\gamma: \; x\cdot 1 + 2x + 3 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \qquad [T(x^2)]_\gamma: \; x\cdot 2x + 4x + 9 = 9 + 4x + 6x^2 = \begin{pmatrix} 9 \\ 4 \\ 2 \end{pmatrix}$$

$$\Rightarrow A = [T]_\gamma = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

- Find eigenvalues of A: calculate the characteristic polynomial:

$$f(t) = \det(A - tI) = \begin{vmatrix} 1-t & 3 & 9 \\ 1 & 3-t & 4 \\ 0 & 0 & 2-t \end{vmatrix} = (2-t)\cdot \begin{vmatrix} 1-t & 3 \\ 1 & 3-t \end{vmatrix} = (2-t)\cdot\left((1-t)(3-t) - 3\right)$$

$$= (2-t)\cdot(t^2 - 4t) = t(t-2)\cdot(t-4) = 0$$

$\Rightarrow \lambda = 0, 2, 4$ are eigenvalues of A

- Calculate eigenvectors:

$\lambda = 0$: $$A - 0\cdot I = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 2$: $$A - 2I = \begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & -9 \\ 0 & 4 & 13 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{13}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

$-9 + 13\cdot\frac{3}{4} = \frac{1}{4}(-36+39)$

$$\Rightarrow v_2' = \begin{pmatrix} -\frac{3}{4} \\ -\frac{13}{4} \\ 1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix}$$ is also an eigenvector.

$$\lambda=4: \quad A-4I=\begin{pmatrix}-3&3&9\\1&-1&4\\0&0&-2\end{pmatrix}\to\begin{pmatrix}1&-1&4\\0&0&1\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&-1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$\Rightarrow v_3=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

Set $\beta=\{f_1(x), f_2(x), f_3(x)\}\subset P_2(\mathbb{R})$ with

$$v_1=\begin{pmatrix}-3\\1\\0\end{pmatrix}=[f_1(x)]_\gamma \Rightarrow f_1(x)=-3+1\cdot x+0\cdot x^2=-3+x.$$

$$v_2=\begin{pmatrix}-3\\-13\\4\end{pmatrix}=[f_2(x)]_\gamma \Rightarrow f_2(x)=-3-13x+4\cdot x^2$$

$$v_3=\begin{pmatrix}1\\1\\0\end{pmatrix}=[f_3(x)]_\gamma \Rightarrow f_3(x)=1+x.$$

Then $[T]_\beta=\begin{pmatrix}0&0&0\\0&2&0\\0&0&4\end{pmatrix}=P^{-1}A\cdot P$ with

$$P=\begin{pmatrix}-3&-3&1\\1&-13&1\\0&4&0\end{pmatrix}$$

<u>Check:</u>

$$T(-3+x)=x\cdot 1+(-1)\cdot x+0=0$$

$$T(-3-13x+4x^2)=x\cdot(-13+8x)+(-3-26+16)\cdot x+(-3-39+36)$$

$$=-6+x\cdot(-13-13)+x^2\cdot 8=-6-26x+8x^2=2\cdot(-3-13x+4x^2)$$

$$T(1+x)=x\cdot 1+3\cdot x+4=4+4x=4\cdot(1+x)$$

✓.

5.2.9 (a) Proof: $T: V \to V$ $\dim V < +\infty$.

T is invertible $\iff$ T is one-to-one

$\iff N(T) = N(T - 0\cdot I) = \{0\}$

$\iff 0$ is not an eigenvalue.

(b) Proof: T invertible $\Rightarrow T$ is one-to-one

$\Rightarrow N(T - 0 I) = N(T) = \{0\}$

$\Rightarrow 0$ is not an eigenvalue of T

We then have the following equivalences:

λ is an eigenvalue of T

$\iff \exists v \neq 0$ s.t. $Tv = \lambda v$

$\iff \exists v \neq 0$, s.t. $v = \lambda \cdot T^{-1} v$

$\iff \exists v \neq 0$ s.t. $T^{-1} v = \lambda^{-1} v$

$\iff \lambda^{-1}$ is an eigenvalue of T^{-1}

20: A: $n\times n$

$$f(t) = \det(A - tI) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

$$\Rightarrow f(0) = \det(A - 0\cdot I) = \det(A)$$

$$= (-1)^n \cdot 0^n + a_{n-1}\cdot 0^{n-1} + \cdots + a_1 0 + a_0 = a_0.$$

A is invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow a_0 \neq 0$. ∎

21. (a) Use induction: $n=1$: $f(t) = \det(A_{11} - t) = A_{11} - t$

$n=2$: $f(t) = \begin{vmatrix} A_{11}-t & A_{12} \\ A_{21} & A_{22}-t \end{vmatrix} = (A_{11}-t)(A_{22}-t) + \underbrace{(-A_{12}\cdot A_{21})}_{q(t)}$ $\deg = 2-2 = 0$.

Assume $(n-1)$-th case is true. For $n\times n$ matrix A,

$$f(t) = \begin{vmatrix} A_{11}-t & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22}-t & \cdots & A_{2n} \\ - & - & - & \\ A_{n1} & A_{n2} & & A_{nn}-t \end{vmatrix} = (A_{11}-t)\cdot \begin{vmatrix} A_{22}-t & \cdots & A_{2n} \\ - & & \cdots \\ A_{n2} & \cdots & A_{nn}-t \end{vmatrix}$$ ← use induction

only $(n-2)$ rows have t variable

$$+ A_{12}\cdot \begin{vmatrix} A_{21} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{33}-t & & A_{3n} \\ A_{n1} & A_{n3} & \cdots & A_{nn}-t \end{vmatrix} + \cdots + (-1)^{1+n} A_{1n}\cdot \begin{vmatrix} A_{21} & A_{22}-t & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33}-t & A_{3n} \\ A_{n1} & A_{n2} & \cdots & A_{n(n-1)} \end{vmatrix}$$

$$= (A_{11}-t)\cdot\left((A_{22}-t)\cdots(A_{nn}-t) + \underset{\uparrow\ \deg\le(n-1)-2 = n-3}{q_1(t)}\right) + A_{12}\cdot \underset{\uparrow\ \deg\le n-2}{g_2(t)} + \cdots + (-1)^{1+n} A_{1n}\cdot \underset{\uparrow\ \deg\le n-2}{g_n(t)}$$

$$= (A_{11}-t)(A_{22}-t)\cdots(A_{nn}-t) + \underset{\uparrow\ \deg\le n-2.}{q(t)}$$ ✓ ∎

5.2.(g): $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$.

· characteristic polynomial:

$-1\cdot(3-t)+1 \qquad (1-t)\cdot(3-t)+1$

$$f(t) = \det(A - tI) = \begin{vmatrix} 3-t & 1 & 1 \\ 2 & 4-t & 2 \\ -1 & -1 & 1-t \end{vmatrix} = \begin{vmatrix} 0 & t-2 & t^2-4t+4 \\ 0 & 2-t & 4-2t \\ -1 & -1 & 1-t \end{vmatrix}$$

$$= -1 \cdot \begin{vmatrix} t-2 & (t-2)^2 \\ -(t-2) & -2(t-2) \end{vmatrix} = (-1)\cdot(t-2)\cdot(-1)(t-2)\cdot \begin{vmatrix} 1 & t-2 \\ 1 & 2 \end{vmatrix}$$

$$= +(t-2)^2 \cdot (2-(t-2)) = -(t-2)^2\cdot(t-4).$$

$\Rightarrow$ eigenvalues $\lambda = 2, 4$.

· Find eigenvectors:

$\lambda = 2$: $A - 2I = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ form a basis for the eigenspace E_2

$\lambda = 4$: $A - 4I = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 4 \\ 0 & -2 & -4 \\ -1 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow v_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

$\Rightarrow Q = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ satisfies $Q^{-1}AQ = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 4 \end{pmatrix}$

<u>Note</u>: $Q = (v_1\ v_2\ \cdots\ v_n)$ satisfies

$$Q^{-1}AQ = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\Longleftrightarrow \quad AQ = Q \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\begin{array}{ccc} AQ & = & Q\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \\ \| & & \| \\ A(v_1\ v_2\ \cdots\ v_n) & & (v_1\ v_2\ \cdots\ v_n) \cdot \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \\ \| & & \| \\ (Av_1\ Av_2\ \cdots\ Av_n) & & (\lambda_1 v_1\ \lambda_2 v_2\ \cdots\ \lambda_n v_n) \end{array}$$

$\Longleftrightarrow \quad Av_i = \lambda_i v_i, \quad i = 1, 2, \cdots, n.$

$\Longleftrightarrow$ The eigenvectors $\{\underset{\lambda_1}{\overset{\updownarrow}{v_1}}, v_2, \cdots, \underset{\lambda_n}{\overset{\updownarrow}{v_n}}\}$ form a basis