!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(15pts) For each of the following subsets of $\mathbf{R}^3$, determine whether it is a vector subspace of $\mathbf{R}^3$. Explain your reason.

(1) $\{(a,b,c); a + 99b = 101c\}$ $=S_1$

(2) $\{(a,b,c); a^2 - b^2 = 0\}$. $=S_2$

(3) $\{(a,b,c); a^2 + b^2 = 0\}$. $=S_3$

(1) Yes. $S_1 = N((1\ 99\ -101))$

$= \{(a,b,c) : (1\ 99\ -101)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\}$.

(5)

(2) No. $S_2 = \{(a+b)(a-b)=0\} = \{a+b=0\} \cup \{a-b=0\}$.

(5)

Not closed under addition e.g. $\underset{S_2}{\underset{\cap}{(1,-1)}} + \underset{S_2}{\underset{\cap}{(1,1)}} = (2,0) \notin S_2$.

(5)

(3) Yes. $S_3 = \{a=b=0\} = \{(0,0,c) : c \in \mathbb{R}\} = \text{Span}\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$.

2(20pts) Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be a linear transformation that satisfies

$$T(1+x) = 1-x, \quad T(x+x^2) = 2+x^2, \quad T(x-x^2) = 2x-x^2.$$

Find the matrix representation of T with respect to the standard basis $\beta = \{1, x, x^2\}$.

$$T(2x) = T((x+x^2)+(x-x^2)) = T(x+x^2) + T(x-x^2)$$

$$= 2+x^2+2x-x^2 = 2+2x$$

$$\Rightarrow T(x) = \frac{1}{2}T(2x) = 1+x$$ ⑤

$$\Rightarrow T(x^2) = T(x+x^2-x) = T(x+x^2) - T(x)$$

$$= 2+x^2-(1+x) = 1-x+x^2 = T(x) - T(x-x^2)$$ ⑤

$$= 1+x-(2x-x^2).$$

$$\Rightarrow T(1) = T(1+x) - T(x)$$ ⑤

$$= 1-x-(1+x) = -2x.$$

$$\Rightarrow [T]_\beta = ([T(1)]_\beta \; [T(x)]_\beta \; [T(x^2)]_\beta)$$ ⑤

$$= \begin{pmatrix} 0 & 1 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

3(25pts) Consider the following subset S of $P_3(\mathbf{R})$.

$$S = \{1 - x, x + x^2, x^2 + x^3, 1 + x^3\}$$

(1) Is S linearly dependent or linearly independent?
(2) Is $x + x^2 + 2x^3$ contained in Span(S)? Explain your reason.

(1) + (2) Choose standard basis $\beta = \{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$. Consider: (5)

$$A = \left([1-x]_\beta \; [x+x^2]_\beta \; [x^2+x^3]_\beta \; [1+x^3]_\beta \,\middle|\, [x+x^2+2x^3]_\beta \right)$$

$$= \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right). \quad (5)$$

(5)

$\Rightarrow$ S is linearly independent (5) $\left(\overset{\#S = 4 = \dim P_3(\mathbb{R})}{\Longrightarrow} S \text{ is a basis for } P_3(\mathbb{R}) \right)$

and $x + x^2 + 2x^3$ is contained in Span$(S) = P_3(\mathbb{R})$ (5)

Check: $-1 \cdot (1-x) + 0 \cdot (x + x^2) + 1 \cdot (x^2 + x^3) + 1 \cdot (1 + x^3)$

$= (-1+1) + x + 1 \cdot x^2 + (1+1)x^3 = x + x^2 + 2x^3$ ✓

4(25pts) Consider the following linear transformation:

$$T : \mathbf{R}^3 \to \mathbf{R}^3, \quad T(a,b,c) = (b+c, a+c, a+b).$$

Determine whether T is diagonalizable. If it is, find a basis β such that $[T]_\beta$ is diagonal.

Let $\alpha = \{e_1, e_2, e_3\}$ be the standard basis

$$[T]_\alpha = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = A \Leftrightarrow T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

A is symmetric $\Rightarrow$ A is diagonalizable. Need find eigenvalues

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 0 & -\lambda^2+1 & \lambda+1 \\ 1 & -\lambda & 1 \\ 0 & 1+\lambda & -\lambda-1 \end{vmatrix} = (-1) \cdot \begin{vmatrix} (1-\lambda)(1+\lambda) & 1+\lambda \\ 1+\lambda & -(\lambda+1) \end{vmatrix}$$

$$= -(\lambda+1)^2 \cdot \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1 \end{vmatrix} = -(\lambda+1)^2 [-1+\lambda-1] = -(\lambda+1)^2 \cdot (\lambda-2)$$

(5)

(5)

$\Rightarrow \lambda = -1,\ m_{-1} = 2$ and $\lambda = 2,\ m_2 = 1$.

$\lambda = -1$: $A - (-1)I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_{-1} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ (5)

$\lambda = 2$: $A - 2I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

(5)

$\Rightarrow E_2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\Rightarrow Q = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ satisfies $Q^{-1}AQ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$ (5)

$\Leftrightarrow \beta = \{(-1,1,0), (-1,0,1), (1,1,1)\} \subset \mathbb{R}^3$ is a basis for $\mathbb{R}^3$ s.t.

$[T]_\beta = D$ is diagonal.

5(25pts) (1) Use the Gram-Schmidt process to $\{1, x, x^2\}$ to find an orthonomal basis β of $P_2(\mathbf{R})$ with respect to the inner product $\langle f, g\rangle = \int_{-2}^{2} f(x)g(x)dx$.
(2) Find the Fourier coefficients of $h(x) = x^2$ with respect to β.

(5)

(1) $V_1 = W_1 = 1$, $\|W_1\|^2 = \int_{-2}^{2} 1\cdot 1 dx = 4 = \|V_1\|^2$

$W_2 = X$, $V_2 = W_2 - \frac{\langle W_2, W_1\rangle}{\|V_1\|^2} V_1$ $\quad \langle W_2, V_1\rangle = \int_{-2}^{2} x\cdot 1 dx = \frac{x^2}{2}\Big|_{-2}^{2} = 0.$

$\Rightarrow V_2 = x - \frac{0}{4} V_1 = x.$ $\quad \|V_2\|^2 = \int_{-2}^{2} x\cdot x\, dx = \frac{x^3}{3}\Big|_{-2}^{2} = \frac{16}{3}.$ (5)

$V_3 = W_3 - \frac{\langle W_3, V_1\rangle}{\|V_1\|^2} V_1 - \frac{\langle W_3, V_2\rangle}{\|V_2\|^2} V_2$

$\langle W_3, V_1\rangle = \int_{-2}^{2} x^2\cdot 1 dx = \frac{x^3}{3}\Big|_{-2}^{2} = \frac{16}{3}$, $\quad \langle W_3, V_2\rangle = \int_{-2}^{2} x^2 x\, dx = \frac{x^4}{4}\Big|_{-2}^{2} = 0.$

(5)

$\Rightarrow V_3 = x^2 - \frac{\frac{16}{3}}{4} 1 = x^2 - \frac{4}{3}$

$\|V_3\|^2 = \int_{-2}^{2} (x^2 - \frac{4}{3})^2 dx = 2\int_0^2 (x^4 - \frac{8}{3}x^2 + \frac{16}{9}) dx = 2\cdot(\frac{x^5}{5} - \frac{8}{9}x^3 + \frac{16}{9}x)\Big|_0^2$

(even fct.)

$= 2\cdot(\frac{32}{5} - \frac{64}{9} + \frac{32}{9}) = 64\cdot(\frac{1}{5} - \frac{1}{9}) = \frac{256}{45} = \frac{16^2}{3^2}\cdot\frac{1}{5}$

(5)

$\Rightarrow u_1 = \frac{V_1}{\|V_1\|} = \frac{1}{2}$, $u_2 = \frac{V_2}{\|V_2\|} = \frac{\sqrt{3}}{4} x$, $u_3 = \frac{V_3}{\|V_3\|} = \frac{3\sqrt{5}}{16}(x^2 - \frac{4}{3})$

$= \sqrt{5}(\frac{3}{16}x^2 - \frac{1}{4})$

form an o.n.b. for $P_2(\mathbb{R})$.

(2) $x^2 = x^2 - \frac{4}{3} + \frac{4}{3}$

$= \frac{16}{3\sqrt{5}} \cdot \frac{3\sqrt{5}}{16}(x^2 - \frac{4}{3}) + \frac{4 \cdot 2}{3} \cdot \frac{1}{2}$

$= \frac{16}{3\sqrt{5}} \cdot u_3 + \frac{8}{3} \cdot u_1$

$= \frac{8}{3} u_1 + 0 \cdot u_2 + \frac{16}{3\sqrt{5}} \cdot u_3$

(5)

Fourier coefficients: $\frac{8}{3}$, 0 , $\frac{16}{3\sqrt{5}} = \frac{16}{15}\sqrt{5}$

$\quad \| \quad\quad \| \quad\quad \|$

$\langle h, u_1 \rangle \quad \langle h, u_2 \rangle \quad \langle h, u_3 \rangle$.

OR: $\langle h, u_1 \rangle = \frac{\langle h, v_1 \rangle}{\|v_1\|} = \frac{16}{3} \cdot \frac{1}{2} = \frac{8}{3}$

$\langle h, v_1 \rangle = \int_{-2}^{2} x^2 \cdot 1 \, dx = \frac{x^3}{3} \Big|_{-2}^{2} = \frac{16}{3}$

$\langle h, u_2 \rangle = \frac{\langle h, v_2 \rangle}{\|v_2\|} = 0$, $\quad \langle h, v_2 \rangle = \int_{-2}^{2} x^2 \cdot x \, dx = \frac{x^4}{4} \Big|_{-2}^{2} = 0$.

$\langle h, u_3 \rangle = \frac{\langle h, v_3 \rangle}{\|v_3\|} = \frac{\frac{256}{45}}{\frac{16}{3} \cdot \frac{1}{\sqrt{5}}} = \frac{16}{15} \cdot \sqrt{5}$

$\langle h, v_3 \rangle = \int_{-2}^{2} x^2 \cdot (x^2 - \frac{4}{3}) \, dx = 2 \cdot \left[\frac{x^5}{5} - \frac{4}{9}x^3\right]_0^2 = 64 \cdot \frac{4}{45} = \frac{256}{45}$

6(20pts) Assume that V is a vector space with a basis $\beta = \{v_1, v_2, v_3, v_4\}$. Let $T : V \to V$ be a linear transformation that satisfies:

$$Tv_1 = v_2, \quad Tv_2 = v_3, \quad Tv_3 = v_4, \quad Tv_4 = 2v_2 + v_3.$$

(1) Calculate the characteristic polynomial of T.
(2) The linear transformation T satisfies the identity:

$$T^4 - T^2 = a \cdot T + b \cdot \mathrm{Id}_V$$

where Id_V is the identity transformation of V. Find the numbers a and b.

(5)

(1). $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = A$

$|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 2 \\ 0 & 1 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} 0 & -\lambda^2 & \lambda+2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$

$= (-\lambda)\cdot(-1)\cdot \begin{vmatrix} -\lambda^2 & \lambda+2 \\ 1 & -\lambda \end{vmatrix} = \lambda \cdot (\lambda^3 - (\lambda+2)) = \lambda^4 - \lambda^2 - 2\lambda$ (5)

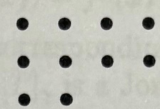
↑ Characteristic polynomial of T.

(2) By Cayley-Hamilton Theorem,

$T^4 - T^2 - 2T = 0 \Leftrightarrow T^4 - T^2 = 2 \cdot T + 0 \cdot \mathrm{Id}_V$ (5)

(5) so $a = 2$, $b = 0$

7(25pts) Let A be a square matrix with characteristic polynomial equal to $(t-2)^{10}$. Assume that the dot diagram of A is the following:



(1) Write down the Jordan canonical form of A.
(2) Calculate $\dim R(A-2I)$ and $\dim R((A-2I)^2)$.

(1) $J(A) = \begin{pmatrix} \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} & & & \\ & \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} & & \\ & & \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} & \\ & & & \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} \end{pmatrix}$ (9)

(2) $\dim N(A-2I) = \#$ dots in 1st. row $= 4$ (4)

$\dim N((A-2I)^2) = \#$ dots in first 2 rows $= 8$ (4)

$\Rightarrow \dim R(A-2I) = 10 - \dim N(A-2I) = 10 - 4 = 6$ (4)

$\dim R((A-2I)^2) = 10 - \dim N((A-2I)^2) = 10 - 8 = 2.$ (4)

8(25pts) Consider the linear transformation:

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R}), \quad T(f(x)) = f''(x) + f'(x) + f(0).$$

(1) Find all eigenvalues of T and the corresponding dot diagrams.
(2) Find a basis γ of $P_2(\mathbf{R})$ such that $[T]_\gamma$ is a Jordan canonical form.

(2)+(1) $\beta = \{1, x, x^2\}$ standard basis for $P_2(\mathbb{R})$.

$$[T]_\beta = \left([T(1)]_\beta \ [T(x)]_\beta \ [T(x^2)]_\beta \right) = \left([1]_\beta \ [1]_\beta \ [2+2x]_\beta \right)$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = A$$ (5)

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) = 0 \Rightarrow \begin{matrix} \lambda = 0, & m_0 = 2 \\ \lambda = 1, & m_1 = 1 \end{matrix}$$ (5)

$$\lambda = 0: \ A - 0 \cdot I = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_0 = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$ (5)

$\dim E_0 = 1 \Rightarrow$ dot diagram for $\lambda = 0$:

• v_1
• v_2

(5)

choose $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ solve: $(A - 0I)v_2 = v_1$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right) \Rightarrow \text{choose } v_2 = \begin{pmatrix} -2 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

(5)

$$\lambda = 1: \ A - 1 \cdot I = \begin{pmatrix} 0 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow E_1 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \bullet v_3$$

$\Rightarrow$ Jordan canonical basis $\gamma = \{ \underset{f_1}{-1+x}, \underset{f_2}{-2+\frac{1}{2}x^2}, \underset{f_3}{1} \}$ satisfies

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Check $T(-1+x) = 0 + 1 + (-1) = 0 \cdot f_1$
$T(-2+\frac{1}{2}x^2) = 1 + x - 2 = x - 1 = 1 \cdot f_1 + 0 \cdot f_2$ ✓
$T(1) = 1 = 1 \cdot f_3$

9(20pts) Find the linear function $f(t) = c_0 + c_1 t$ that has the best fit to the data:

$$\{(0,0),(1,2),(2,1),(3,-1),(4,0)\} = \{(t_i, y_i); 1 \le i \le 5\}.$$

with respect to the error: $\mathbf{E} = \sum_{i=1}^{5}(c_0 + c_1 t_i - y_i)^2$.

Set $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$ $\quad y = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & \underset{\substack{\uparrow \\ 1+4+9+16}}{30} \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix}.$$

$$(A^TA)^{-1} = \frac{1}{5} \cdot \frac{1}{6-4}\begin{pmatrix} 6 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{10} \cdot \begin{pmatrix} 6 & -2 \\ -2 & 1 \end{pmatrix}.$$

$$A^Ty = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 6 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{3}{10} \end{pmatrix}$$

$\Rightarrow f(t) = 1 - \frac{3}{10}t$ has the best fit to the data.