Goal:

Main Thm: Let A be an $n \times n$ symmetric matrix over $\mathbb{R}$: $A^T = A$.

Then there exists an orthogonal matrix S s.t. $\underset{\overset{\|}{S^TAS}}{S^{-1}AS}$ is diagonal.

· Def: An $n \times n$ matrix S is orthogonal if $S^TS = I_n$

Prop: The following conditions are equivalent to each other for an $n \times n$ matrix:

1. S is orthogonal i.e. $S^TS = I_n$
2. $S^{-1} = S^T$
3. $S \cdot S^T = I_n$
4. If $S = (u_1\ u_2 \cdots u_n)$, then $\{u_1, \cdots, u_n\}$ form an o.n.b. for $\mathbb{R}^n$.

Proof: $S^TS = I_n \Leftrightarrow S^{-1} = S^T \Leftrightarrow S \cdot S^T = I_n$ (because S, S^T are square matrices)

To see $1. \Leftrightarrow 4.$, we observe:

$$S^T \cdot S = \begin{pmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{pmatrix} (u_1\ u_2 \cdots u_n) = \begin{pmatrix} u_1^Tu_1 & u_1^Tu_2 & \cdots & u_1^Tu_n \\ u_2^Tu_1 & u_2^Tu_2 & \cdots & u_2^Tu_n \\ - & - & - & - \\ u_n^Tu_1 & u_n^Tu_2 & \cdots & u_n^Tu_n \end{pmatrix}$$

$$= \begin{pmatrix} \|u_1\|^2 & \langle u_1, u_2 \rangle & \cdots & \langle u_1, u_n \rangle \\ \langle u_2, u_1 \rangle & \|u_2\|^2 & \cdots & \langle u_2, u_n \rangle \\ - & - & - & - \\ \langle u_n, u_1 \rangle & \langle u_n, u_2 \rangle & \cdots & \|u_n\|^2 \end{pmatrix}$$

So $S^TS = I_n \Leftrightarrow \begin{cases} \|u_1\|^2 = \cdots = \|u_n\|^2 = 1 \\ \langle u_i, u_j \rangle = 0 \text{ if } i \neq j \end{cases} \Leftrightarrow \{u_1, u_2, \cdots, u_n\}$ is an o.n.b. for $\mathbb{R}^n$.

To achieve our goal, we prove two statements:

<u>Prop I</u>: If A is a real symmetric matrix, then $f(\lambda)=|A-\lambda I|$ splits $/\mathbb{R}$.

<u>Prop II</u>: If $f(\lambda)=|A-\lambda I|$ splits over $\mathbb{R}$, then there is an orthogonal matrix S such that $S^TAS=S^{-1}AS$ is upper triangular.

Prop I + Prop II $\Rightarrow$ <u>Main Thm</u>:

$$(S^TAS)^T=S^TA^T(S^T)^T \overset{A^T=A}{=\!=\!=} S^TAS$$

symmetric upper triangular matrices are just diagonal matrices!

Proof of <u>Prop I</u>: $f(\lambda)=|A-\lambda I|$ always splits over $\mathbb{C}$ (fundamental thm of algebra)

Let $\lambda_1\in\mathbb{C}$ be any root to $f(\lambda)$. Then $\exists\ \underset{0}{\underset{\neq}{v}}\in\mathbb{C}^n$ s.t. $Av=\lambda_1 v$.

$$\Rightarrow \underset{v^*A^*}{\underset{\parallel}{(Av)^*}}=\underset{v^*\bar{\lambda}_1}{\underset{\parallel}{(\lambda_1 v)^*}}$$

(for any complex matrix B, $B^*=\bar{B}^T$ (conjugated transpose) $\parallel$ (transposed conjugation))

$\Rightarrow v^*A^*v=\bar{\lambda}_1 v^*v$ If A is real symmetric, then $A^*=\bar{A}^T=A^T=A$

$\Rightarrow v^*A\cdot v=\bar{\lambda}_1 v^*v$. On the other hand $Av=\lambda_1 v\Rightarrow v^*Av=\lambda_1 v^*v$.

$\Rightarrow \bar{\lambda}_1 v^*v=\lambda v^*v$ i.e. $(\lambda_1-\bar{\lambda}_1)v^*v=(\lambda_1-\bar{\lambda}_1)\|v\|^2=0 \xRightarrow{v\neq 0} \lambda_1-\bar{\lambda}_1=0 \Updownarrow \lambda_1$ is real. ∎

$$\left(v^*\cdot v=(a_1-ib_1\ \cdots\ a_n-ib_n)\begin{pmatrix}a_1+ib_1\\ \vdots\\ a_n+ib_n\end{pmatrix}=(a_1-ib_1)(a_1+ib_1)+\cdots+(a_n-ib_n)(a_n+ib_n)=(a_1^2+b_1^2)+\cdots+(a_n^2+b_n^2)=\|v\|^2\neq 0\right)$$

Proof of Prop II: $f(\lambda) = |A - \lambda I|$ splits $/\mathbb{R}$

$\Longrightarrow \exists\, Q = (w_1\ w_2 \cdots w_n)$ s.t. $Q^{-1}AQ$ is upper triangular

(or even Jordan form)

$$\Rightarrow \begin{cases} Aw_1 \in \mathrm{Span}\{w_1\} \\ Aw_2 \in \mathrm{Span}\{w_1, w_2\} \\ \cdots\cdots \\ Aw_j \in \mathrm{Span}\{w_1, w_2, \cdots, w_j\} \\ \cdots \end{cases}$$

Use Gram-Schmidt $\{w_1, w_2, \cdots, w_n\} \xrightarrow{\text{o.n.b.}} \{u_1, u_2, \cdots, u_n\}$.

Then $\mathrm{Span}\{w_1, \cdots, w_j\} = \mathrm{Span}\{u_1, u_2, \cdots, u_j\}$, $1 \le j \le n$.

$$\Rightarrow \begin{cases} Au_1 \in \mathrm{Span}\{u_1\} \\ \cdots\cdots \\ Au_j \in \mathrm{Span}\{u_1, \cdots, u_j\} \end{cases}$$

$\Rightarrow S = (u_1\ u_2 \cdots u_n)$ is an orthogonal matrix that satisfies

$S^T A S = S^{-1} A S$ is upper triangular. ∎

Q: How to find S in practice for a symmetric matrix A.

Ans:

Step 1: Find (distinct) eigenvalues $\lambda_1, \cdots, \lambda_k$ for A.

(We know that $f(\lambda) = (\lambda-\lambda_1)^{m_1} \cdots (\lambda-\lambda_k)^{m_k}$ splits).

Step 2: For each λ_j, find eigenspace

$E_{\lambda_j} = \mathrm{Span}\{w_1^{(j)}, \cdots, w_{m_j}^{(j)}\}$ (we know $\dim E_{\lambda_j} = m_j$ since A is diagonalizable)

Step 3: Use Gram-Schmidt for just $\{w_1^{(j)}, \cdots, w_{m_j}^{(j)}\}$

to get o.n.b. $\beta_j = \{u_1^{(j)}, \cdots, u_{m_j}^{(j)}\}$ for E_{λ_j}.

Step 4: $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ produces S:

$$S = \left(u_1^{(1)} \cdots u_{m_1}^{(1)} \; u_1^{(2)} \cdots u_{m_2}^{(2)} \cdots \; u_1^{(k)} \cdots u_{m_k}^{(k)} \right)$$

(Just collect o.n.b. for E_{λ_j}, $1 \leq j \leq n$ together in order to form an o.n.b. for $\mathbb{R}^n$ that produces an orthogonal matrix S satisfying $S^T A S$ is diagonal.

Ex: $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = A$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 0 & (1-\lambda)(\lambda-1)+1 & 2 \\ 1 & 1-\lambda & 0 \\ 0 & 2(\lambda-1) & 1-\lambda \end{vmatrix} = -1\cdot(\lambda-1)\begin{vmatrix} -\lambda^2+2\lambda & 2 \\ 2 & -1 \end{vmatrix}$$

$$= -(\lambda-1)\cdot(\lambda^2-2\lambda-4) = -(\lambda-1)(\lambda-a_1)(\lambda-a_2)$$

$$a_1, a_2 = \frac{2 \pm \sqrt{4+16}}{2} = 1 \pm \sqrt{5}$$

$$\Rightarrow \lambda = 1,\ 1+\sqrt{5},\ 1-\sqrt{5}.$$

$\lambda = 1$: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_1 = \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}_{w_1} \right\}$

$\lambda = 1+\sqrt{5}$: $\begin{pmatrix} -\sqrt{5} & 1 & 2 \\ 1 & -\sqrt{5} & 0 \\ 2 & 0 & -\sqrt{5} \end{pmatrix} \to \begin{pmatrix} 1 & -\sqrt{5} & 0 \\ 0 & -4 & 2 \\ 0 & 2\sqrt{5} & -\sqrt{5} \end{pmatrix} \to \begin{pmatrix} 1 & -\sqrt{5} & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Leftrightarrow \begin{cases} x_1 - \sqrt{5}x_2 = 0 \\ 2x_2 - x_3 = 0 \end{cases} \Rightarrow E_{1+\sqrt{5}} = \mathrm{Span}\left\{ \overbrace{\begin{pmatrix} \sqrt{5} \\ 1 \\ 2 \end{pmatrix}}^{w_2} \right\}.$

$\lambda = 1-\sqrt{5}$: $\begin{pmatrix} \sqrt{5} & 1 & 2 \\ 1 & \sqrt{5} & 0 \\ 2 & 0 & \sqrt{5} \end{pmatrix} \to \begin{pmatrix} 1 & \sqrt{5} & 0 \\ 0 & -4 & 2 \\ 0 & -2\sqrt{5} & \sqrt{5} \end{pmatrix} \to \begin{pmatrix} 1 & \sqrt{5} & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_{1-\sqrt{5}} = \mathrm{Span}\left\{ \overbrace{\begin{pmatrix} -\sqrt{5} \\ 1 \\ 2 \end{pmatrix}}^{w_3} \right\}$

Note that $\{w_1, w_2, w_3\}$ automatically is an orthogonal basis

$$\Rightarrow \quad u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{10}}\begin{pmatrix} \sqrt{5} \\ 1 \\ 2 \end{pmatrix}, \quad \underset{\frac{w_3}{\|w_3\|}}{\underset{\|}{u_3}} = \frac{1}{\sqrt{10}}\begin{pmatrix} -\sqrt{5} \\ 1 \\ 2 \end{pmatrix}$$

form an o.n.b. for $\mathbb{R}^3$ and

$$S = (u_1\ u_2\ u_3) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{pmatrix}$$ is an orthogonal matrix

that diagonalizes A : $$S^T A S = \begin{pmatrix} 1 & & \\ & 1+\sqrt{5} & \\ & & 1-\sqrt{5} \end{pmatrix}$$

For more examples that uses Gram-Schmidt for eigenspaces

See HW problem 2(d), 2(e).