Inner product space (over real numbers)

V a vector space$/\mathbb{R}$

Inner product: $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ satisfies

$(v, w) \mapsto \langle v, w\rangle$

(i) $\langle c_1 v_1 + c_2 v_2 , w\rangle = c_1 \langle v_1, w\rangle + c_2 \langle v_2 , w\rangle$

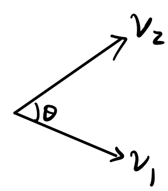
(ii) $\langle v, w\rangle = \langle w, v\rangle$

(iii) $\langle v, v\rangle \geq 0$, and >0 if $v \neq 0$. $\left(\|v\|^2 = \langle v, v\rangle \geq 0 \right)$

Standard Example: $V = \mathbb{R}^n$. $v_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $v_2 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

$\langle v_1 , v_2 \rangle = v_1^T v_2 = (a_1 \cdots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$

$\|v_1\|^2 = \langle v_1 , v_1 \rangle = a_1^2 + a_2^2 + \cdots + a_n^2$.



Geometrically: $\langle v_1, v_2\rangle = \|v_1\| \cdot \|v_2\| \cdot \cos\theta$

Pythagorean thm: In any inner product space V, if $\langle v_1, v_2 \rangle = 0$

$\Updownarrow$

$v_1 \perp v_2$

$\Updownarrow$

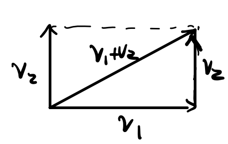
v_1 is orthogonal to v_2

Then $\|v_1 + v_2\|^2 = \langle v_1 + v_2 , v_1 + v_2 \rangle$

$= \langle v_1, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_1\rangle + \langle v_2, v_2\rangle$

$= \|v_1\|^2 + 0 + 0 + \|v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$

Geometrically.



Def: $S = \{v_1, v_2, \cdots, v_r\} \subset V$ is an orthogonal subset if

$\langle v_i, v_j \rangle = 0$ for any $i \neq j$.

S is orthonormal (o.n.) if $\|v_i\| = 1$ and $\langle v_i, v_j \rangle = 0$, $i \neq j$.

S is an orthonormal basis (o.n.b.) if S is orthonormal and is a basis.

Fact: Any orthonormal subset S is linearly independent.

Pf: $v = c_1 v_1 + c_2 v_2 + \cdots + c_r v_r = 0$

$\Rightarrow 0 = \langle v, v_i \rangle = \langle \sum_{j=1}^{r} c_j v_j, v_i \rangle = c_i \langle v_i, v_i \rangle = 0. \ \forall i.$ ∎

- Assume $S = \{w_1, w_2, \cdots, w_r\}$ is any subset of V.

Gram-Schmidt othogonalization process produce an orthonormal basis (o.n.b.) for $\mathrm{Span}(S)$:

Set $v_1 = w_1 \rightsquigarrow u_1 = \frac{v_1}{\|v_1\|}$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = w_2 - \langle w_2, u_1 \rangle u_1 \rightsquigarrow u_2 = \frac{v_2}{\|v_2\|}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= w_3 - \langle w_3, u_1 \rangle u_1 - \langle w_3, u_2 \rangle u_2 \rightsquigarrow u_3 = \frac{v_3}{\|v_3\|}$$

$$= w_3 - \underbrace{\left(\langle w_3, u_1 \rangle u_1 + \langle w_3, u_2 \rangle u_2 \right)}_{\mathrm{Proj}_{\mathrm{Span}\{u_1, u_2\}} w_3}$$

In general, assume $v_1, \cdots, v_{k-1}$ are obtained, set

$$v_k = w_k - \frac{\langle w_k, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_k, v_2\rangle}{\|v_2\|^2} v_2 - \cdots - \frac{\langle w_k, v_{k-1}\rangle}{\|v_{k-1}\|^2} v_{k-1}$$

$$\rightsquigarrow \quad u_k = \frac{v_k}{\|v_k\|}.$$

$$\mathrm{Span}\{w_1, w_2, \cdots, w_k\} = \mathrm{Span}\{v_1, v_2, \cdots, v_k\}$$
$$= \mathrm{Span}\{u_1, u_2, \cdots, u_k\}$$

$$v_k = w_k - \mathrm{Proj}_{\mathrm{Span}\{u_1, u_2, \cdots, u_k\}} w_k$$
$$= w_k - \sum_{j=1}^{k-1} \langle w_k, u_j\rangle u_j = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j\rangle}{\|v_j\|^2} v_j$$

If $\alpha = \{w_1, \cdots, w_n\}$ is a basis, then Gram-Schmidt produces an o.n.b..

$$\alpha \rightsquigarrow \underbrace{\{v_1, v_2, \cdots, v_n\}}_{\text{orthogonal subset}} \rightsquigarrow \{u_1, u_2, \cdots, u_n\}$$

$$u_1 = \frac{v_1}{\|v_1\|}, \quad u_2 = \frac{v_2}{\|v_2\|}, \quad \ldots, \quad u_n = \frac{v_n}{\|v_n\|}$$

Any $v \in V$: $\quad v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$ with the coefficients given by

$a_i = \langle v, u_i\rangle$: called Fourier coefficients w.r.t. the o.n.b. $\{u_1, \cdots, u_n\}$.

$$a_i = \left\langle v, \frac{v_i}{\|v_i\|}\right\rangle = \frac{\langle v, v_i\rangle}{\|v_i\|}$$

Example: $S = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ with $w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $w_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $w_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$

$$v_1 = w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \|v_1\|^2 = 1^2+1^2+1^2 = 3$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{6}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}. \qquad \|v_2\|^2 = (-1)^2 + 0^2 + 1^2 = 2.$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{0}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}.$$

Check $\langle v_2, v_1\rangle = 0$
$\langle v_3, v_1\rangle = 0$ ✓
$\langle v_3, v_2\rangle = 0$

$$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}.$$

form an o.n.b.

Find Fourier coefficients for $v = \begin{pmatrix}2\\3\\4\end{pmatrix}$: $v = a_1 u_1 + a_2 u_2 + a_3 u_3$

$$a_1 = \langle v, u_1\rangle = \frac{\langle v, v_1\rangle}{\|v_1\|} = \frac{9}{\sqrt{3}}$$

$$a_2 = \langle v, u_2\rangle = \frac{\langle v, v_2\rangle}{\|v_2\|} = \frac{2}{\sqrt{2}}$$

$$a_3 = \langle v, u_3\rangle = \frac{\langle v, v_3\rangle}{\|v_3\|} = \frac{\frac{1}{3}(2\cdot 1 + 3\cdot(-2) + 4\cdot 1)}{\frac{1}{3}\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|} = 0$$

$$\Rightarrow v = \frac{9}{\sqrt{3}} u_1 + \frac{2}{\sqrt{2}} u_2 \in \operatorname{Span}\{u_1, u_2\} = \operatorname{Span}\{w_1, w_2\}$$

(Indeed: $v = w_1 + w_2$)

Ex: $V = P_2(\mathbb{R})$, $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$, $\forall f, g \in P_2(\mathbb{R})$

$$\alpha = \{\underset{\overset{\shortparallel}{w_1}}{x^2}, \underset{\overset{\shortparallel}{w_2}}{x}, \underset{\overset{\shortparallel}{w_3}}{1}\}$$

$v_1 = w_1 = x^2$. $\quad \|v_1\|^2 = \int_0^1 x^2 \cdot x^2\,dx = \frac{1}{5}x^5\Big|_0^1 = \frac{1}{5}$.

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 \qquad \langle w_2, v_1\rangle = \int_0^1 x \cdot x^2\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}$$

$$= x - \frac{\frac{1}{4}}{\frac{1}{5}} x^2 = x - \frac{5}{4}x^2.$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$\langle w_3, v_1\rangle = \int_0^1 1 \cdot x^2\,dx = \frac{1}{3}. \quad \langle w_3, v_2\rangle = \int_0^1 1 \cdot (x - \tfrac{5}{4}x^2)\,dx = \frac{1}{2} - \frac{5}{12} = \frac{1}{12}.$$

$$\|v_2\|^2 = \langle v_2, v_2\rangle = \int_0^1 (x - \tfrac{5}{4}x^2)\cdot(x - \tfrac{5}{4}x^2)\,dx = \int_0^1 (x^2 - \tfrac{5}{2}x^3 + \tfrac{25}{16}x^4)\,dx$$

$$= \frac{1}{3} - \frac{5}{8} + \frac{5}{16} = \frac{1}{48}\cdot(16 - 30 + 15) = \frac{1}{48}.$$

$$\Rightarrow v_3 = 1 - \frac{\frac{1}{3}}{\frac{1}{5}}x^2 - \frac{\frac{1}{12}}{\frac{1}{48}}(x - \tfrac{5}{4}x^2) = 1 - \frac{5}{3}x^2 - 4x + 5x^2 = 1 - 4x + \frac{10}{3}x^2$$

$$\|v_3\|^2 = \langle v_3, v_3\rangle = \int_0^1 (1 - 4x + \tfrac{10}{3}x^2)^2\,dx = \int_0^1 (1 + 16x^2 + \tfrac{100}{9}x^4 - 8x + \tfrac{20}{3}x^2 - \tfrac{80}{3}x^3)\,dx$$

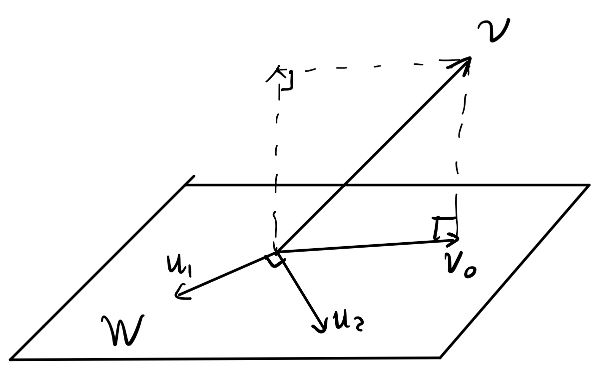
$$= 1 + \frac{16}{3} + \frac{20}{9} - 4 + \frac{20}{9} - \frac{20}{3} = \frac{1}{9}$$

$$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \sqrt{5}\cdot x^2, \quad u_2 = \frac{v_2}{\|v_2\|} = \sqrt{48}\cdot\left(x - \frac{5}{4}x^2\right) = \sqrt{3}\cdot(4x - 5x^2)$$

$$u_3 = \frac{v_3}{\|v_3\|} = 3\cdot\left(1 - 4x + \frac{10}{3}x^2\right) = 3 - 12x + 10x^2$$

form an orthonormal basis for $P_2(\mathbb{R})$.

- $W \subseteq V$ subspace



$$v_0 = \mathrm{Proj}_W v = \sum_{i=1}^{\dim W} \langle v, u_i \rangle u_i$$

v_0 is the vector in W that is "closest to v".

v_0 satisfies $v - v_0 \perp W$

Application: Optimal approximation