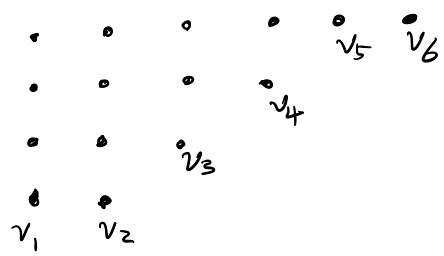
- $T: V \to V$ has a Jordan canonical form
  $\Leftrightarrow$ characteristic polynomial splits.

- Diagonal entries of the Jordan canonical form
  $=$ eigenvalues of $T$

- Let $\lambda$ be an eigenvalue of $T$.
  Dot diagram associated to $\lambda$ $\Leftrightarrow$ Jordan blocks associated to $\lambda$ (corresponding to)

Ex:



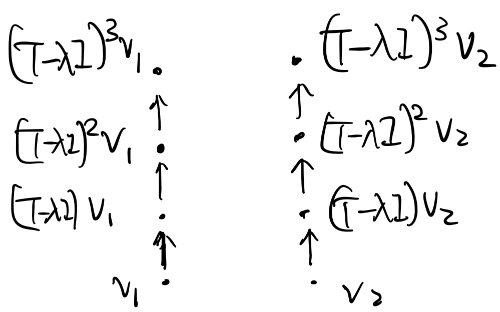

$\#$ dots on 1st row $= \#$ of columns $= \dim N(T-\lambda I)$.

$\#$ dots on $k$-th row $= \dim N((T-\lambda I)^k) - \dim N((T-\lambda I)^{k-1})$.

How to find a Jordan Canonical basis:

1. Find $v_1, v_2 \in N((T-\lambda I)^4) - N((T-\lambda I)^3)$ s.t.
   $v_1$ and $v_2$ are linearly indep. (modulo $N((T-\lambda I)^3)$).

$\rightsquigarrow$ recover the 1st. and 2nd column:



2. Find $v_3 \in N((T-\lambda I)^3) - N((T-\lambda I)^2)$ s.t.

$v_3$, $(T-\lambda I)v_1$, $(T-\lambda I)v_2$ are linearly independent (modulo $N(T-\lambda I)^2$)

recover the 3rd column:

$$\begin{array}{l} 0 \\ \uparrow \\ \cdot\ (T-\lambda I)^2 v_3 \\ \uparrow \\ \cdot\ (T-\lambda I) v_3 \\ \uparrow \\ \cdot\ v_3 \end{array}$$

3. Find $v_4 \in N((T-\lambda I)^2) - N(T-\lambda I)$ s.t.

$v_4$, $(T-\lambda I)v_3$, $(T-\lambda I)^2 v_1$, $(T-\lambda I)^2 v_2$ are linearly independent (modulo $N(T-\lambda I)$)

recover the 4-th column:

$$\begin{array}{l} 0 \\ \uparrow \\ \cdot\ (T-\lambda I) v_4 \\ \uparrow \\ \cdot\ v_4 \end{array}$$

4. Find $v_5, v_6 \in N(T-\lambda I)$ s.t.

$v_5, v_6, (T-\lambda I)v_4, (T-\lambda I)^2 v_3, (T-\lambda I)^3 v_1, (T-\lambda I)^3 v_2$ are linearly independent.

$\rightsquigarrow$ Find disjoint cycles of generalized vectors that form a basis for the space of generalized eigenvectors corresponding to $\lambda$

$K_\lambda = \{v \in V : (T-\lambda I)^p v = 0 \text{ for some } p \geq 1\}$.

Ex: $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$, Calculate $A^{10}$

- $f(t) = \begin{vmatrix} 1-t & 1 \\ -1 & 3-t \end{vmatrix} = (1-t)(3-t)+1 = t^2-4t+4 = (t-2)$

$\Rightarrow$ eigenvalue $\lambda = 2$.

- $A-2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow N(A-2I) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

$\Rightarrow$ dot diagram: $\begin{array}{l} 0 \\ \uparrow \\ \bullet\, v_2 \\ \uparrow \\ \bullet\, v_1 \end{array}$

- $(A-2I)^2 = O$ (by Caley-Hamilton Thm.) $\Rightarrow N((A-2I)^2) = \mathbb{R}^2$

Choose $v_1 \in \mathbb{R}^2 - N(A-2I)$: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow v_2 = (A-2I)v_1 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

$\Rightarrow Q = (v_2\ v_1) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ satisfies $Q^{-1}AQ = J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

$\Rightarrow A = Q\cdot J\cdot Q^{-1} \Rightarrow A^{10} = QJQ^{-1}\cdot QJQ^{-1} \cdots QJQ^{-1}\cdot QJQ^{-1}$

$= Q\cdot J^{10}\cdot Q^{-1}$.

- Calculate $J^{10}$: $J = 2\cdot I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^k = 0$ if $k \geq 2$.

binomial formula: If $AB=BA$, then

$$(A+B)^p = \sum_{k=0}^{p} \binom{p}{k} A^k \cdot B^{p-k}$$

$$= A^p + p \cdot A^{p-1} \cdot B + \frac{p(p-1)}{2} A^{p-2} \cdot B^2 + \cdots + \frac{p(p-1)}{2} A^2 \cdot B^{p-2} + p \cdot A B^{p-1} + B^p$$

$$= A^p + p \cdot A^{p-1} \cdot B + B^2 C.$$

So $J^{10} = (2I)^{10} + 10 (2I)^{10-1} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0$

$$= 2^{10} \cdot I + 5 \cdot 2^{10} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2^{10} \cdot \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}.$$

$$\Rightarrow A^{10} = Q \cdot J^{10} \cdot Q^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \cdot 2^{10} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$$

$$= 2^{10} \cdot \begin{pmatrix} -1 & -4 \\ -1 & -5 \end{pmatrix} \cdot \frac{1}{1} \cdot \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= 2^{10} \cdot \begin{pmatrix} -4 & 5 \\ -5 & 6 \end{pmatrix} = 1024 \cdot \begin{pmatrix} -4 & 5 \\ -5 & 6 \end{pmatrix}$$