- $T: V \longrightarrow V$ linear transformation
- β: a basis for V. $\rightsquigarrow$ matrix representation $A = [T]_\beta$
- characteristic polynomial of T:

  $f(t) = \det(A - tI)$.

- Assume T has a Jordan canonical form:

  $\exists$ a basis $\gamma = \{v_1, \dots, v_n\}$ s.t.

$$[T]_\gamma = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_r \end{pmatrix} \quad \text{where} \quad C_i = \left.\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}\right\} d_i$$

$\rightsquigarrow$ characteristic polynomial of T

$$f(t) = \det([T]_\gamma - tI) = \det(C_1 - tI)\det(C_2 - tI)\cdots\det(C_r - tI)$$

$$= (\lambda_1 - t)^{d_1}(\lambda_2 - t)^{d_2}\cdots(\lambda_r - t)^{d_r}$$

$\Rightarrow$ $f(t)$ splits. and eigenvalues are exactly diagonal entries.

<u>Conversely</u>: $f(t)$ splits $\Rightarrow$ $\exists$ Jordan canonical form.

Consider C_1 : $\rightsquigarrow$ a T-invariant subspace $W = \mathrm{Span}\underbrace{\{v_1, \dots, v_{d_1}\}}_{\gamma'}$

s.t. $$[(T-\lambda_1 I)|_W]_{\gamma'} = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

$\iff$ $(T-\lambda_1 I)(v_1) = 0$

$(T-\lambda_1 I)(v_2) = v_1$

$\vdots$

$(T-\lambda_1 I)^{d_1-1}(v_{d_1}) = v_{d_1-1}$

0

$\uparrow$

$\bullet\ v_1 = (T-\lambda_1 I)v_2$

$\uparrow$

$\bullet\ v_2$

$\vdots$

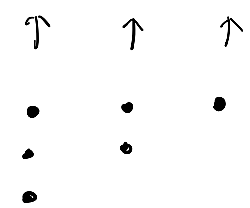
$\bullet\ v_{d_1-1} = (T-\lambda_1 I)(v_{d_1})$

$\uparrow$

$\bullet\ v_{d_1}$

Each Jordan block corresponds to a cycle of length d.
of size d

Ex: $$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}} & & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \\ & & \boxed{2} \end{pmatrix}$$



dot diagram

- # of cycles associated to an eigenvalue λ

$= \dim N(A-\lambda I)$.

- # of dots on the k-row (corresp. to λ)

$= \dim N((A-\lambda I)^k) - \dim N((A-\lambda I)^{k-1})$.

- total # of dots (corresponding to λ)

$= \underbrace{\dim K_\lambda}_{\text{space of generalized eigenvectors}} =$ multiplicity of λ

$K_\lambda = \{ v \in V : (A-\lambda I)^p v = 0 \text{ for some } p \geq 1 \}$.

$\rightsquigarrow \sum_\lambda \dim K_\lambda = \sum_\lambda \text{mult}(\lambda) = n$.

Eb: $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$

- $f(t) = \begin{vmatrix} 2-t & -1 & 0 & 1 \\ 0 & 3-t & -1 & 0 \\ 0 & 1 & 1-t & 0 \\ 0 & -1 & 0 & 3-t \end{vmatrix} = (2-t) \cdot \begin{vmatrix} 3-t & -1 & 0 \\ 1 & 1-t & 0 \\ -1 & 0 & 3-t \end{vmatrix}$

$$= (2-t)\cdot(3-t)\cdot \begin{vmatrix} 3-t & -1 \\ 1 & 1-t \end{vmatrix} = (2-t)\cdot(3-t)\Big[\underbrace{(3-t)(1-t)+1}_{\substack{\parallel \\ t^2-4t+4=(t-2)^2}}\Big]$$

$$= (2-t)^3 \cdot (3-t).$$

$\Rightarrow \lambda = 2, 3.$

- $\lambda = 3$: $\quad mult(3) = 1 = \underset{\substack{\dim(K_3) \\ \parallel}}{\dim(E_3)} \Rightarrow$ • $\quad$ 1 dot

- $\lambda = 2$: $\quad mult(2) = 3 = \dim(K_3 \Rightarrow$ 3 dots.

To determine the dot diagram and to find Jordan basis we need to compute $N(A-2I)$ and $N((A-2I)^2)$.

$$A-2I = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow N(A-2I) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \quad \dim N(A-2I) = 2.$$

$\Rightarrow$ 2 cycles $\Rightarrow$ dot diagram

v_1 • • v_3

v_2 •

$$(A-2I)^2 = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow N((A-2I)^2) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}}_{\text{in } N((A-2I)^2) \text{ but not in } N(A-2I)} \right\}.$$

$\Rightarrow$ choose $v_2 \in N((A-2I)^2) - N(A-2I)$. for example

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}.$$

$$\rightsquigarrow \quad v_1 = (A-2I)v_2 = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$\rightsquigarrow$ Choose $v_3 \in N(A-2I)$ that is linearly indep. w. v_1

$\rightsquigarrow$ Choose $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\Rightarrow \quad \begin{tikzcd} 0 & 0 \\ v_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \bullet \arrow[u] & \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \arrow[u] \\ v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \bullet \arrow[u] & \end{tikzcd}$$

eigenvector for $\lambda = 3$

$$\Rightarrow Q = \begin{pmatrix} u & v_1 & v_2 & v_3 \end{pmatrix} \text{ satisfies}$$

$$Q^{-1}AQ = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & \\ & \boxed{2} & \\ & & \boxed{3} \end{pmatrix}.$$