- A $n \times n$ diagonalizable

$\Leftrightarrow \exists Q$ s.t. $Q^{-1} \cdot A \cdot Q = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$.

$\Leftrightarrow \begin{cases} f(t) = \det(A - tI) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k} \quad \text{(splits)} \\ \dim E_{\lambda_i} = \text{mult}(\lambda_i) \quad i = 1, \cdots, k. \end{cases}$

- In general, if $f(t)$ splits, then

$\exists Q$ invertible s.t. $Q^{-1} \cdot A \cdot Q = J = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_r \end{pmatrix}$

Jordan canonical form

each C_i is a Jordan block:

$$C_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

Jordan block of size 1: (λ)

2: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

3: $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

4: $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$

Ex:
$$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & & \\ & \boxed{2} & & 0 & \\ & & \boxed{\begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}} & & \\ & 0 & & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix} \underset{\text{different}}{\not\Leftrightarrow} \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}} & & \\ & \boxed{\begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}} & \\ & & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

same characteristic polynomial
$$(2-t)^3 \cdot (3-t)^2 \cdot (0-t)^2 = (2-t)^3 (3-t)^2 \cdot t^2$$

Ex: $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

- $f(t) = \begin{vmatrix} 1-t & 1 \\ -1 & 3-t \end{vmatrix} = t^2 - 4t + 4 = (t-2)^2 \Rightarrow \lambda = 2, \ \text{mult}(2) = 2$
- $A - 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ E_2 = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$\dim E_2 = 1 < 2 = \text{mult}(2) \Rightarrow$ not diagonalizable

$\Rightarrow$ Jordan canonical form $= \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$\Rightarrow \exists\, Q$ invertible s.t. $Q^{-1} A \cdot Q = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

- Find $Q = (v_1 \ v_2)$

$$\begin{array}{ccc} AQ & = & Q\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \\ \| & & \| \\ A(v_1 \ v_2) & & (v_1 \ v_2)\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \\ \| & & \| \\ (Av_1 \ Av_2) & = & (2v_1 \quad v_1 + 2v_2) \end{array}$$

$$\Leftrightarrow \begin{cases} Av_1 = 2v_1 \\ Av_2 = v_1 + 2v_2 \end{cases} \Leftrightarrow \begin{cases} (A-2I)v_1 = 0 \\ (A-2I)v_2 = v_1 \end{cases}$$

$\Rightarrow$ a cycle (chain) of length 2:

$$\begin{tikzcd} 0 \\ v_1 \arrow[u] \\ v_2 \arrow[u, "A-2I"'] \end{tikzcd} \qquad \begin{tikzcd} 0 \\ \bullet \arrow[u] \\ \bullet \arrow[u] \end{tikzcd}$$

Find v_1: v_1 is an eigenvector for $\lambda=2$:

we can choose $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as above.

Find v_2: v_2 is a solution to $(A-2I)v_2 = v_1$

$$\Rightarrow \underset{\text{augmented matrix}}{(A-2I \mid v_1)} = \left(\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ any solution e.g. $v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

$\Rightarrow$ $Q = (v_1 \ v_2) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ satisfies $Q^{-1} A Q = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

- In general, a Jordan block of size d corresponds to a cycle of length d:

$$\underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{d} \longleftrightarrow \begin{tikzcd} 0 \\ (A-\lambda I)^{d-1} v \arrow[u, "A-\lambda I"'] \\ \vdots \arrow[u] \\ (A-\lambda I) v \arrow[u, "A-\lambda I"'] \\ v_1 \arrow[u, "A-\lambda I"'] \end{tikzcd} \qquad \begin{tikzcd} 0 \\ \bullet \arrow[u] \\ \vdots \arrow[u] \\ \bullet \arrow[u] \\ \bullet \arrow[u] \end{tikzcd}$$

d dots in the dot diagram

$$\gamma = \{ (A-\lambda I)^{d-1} v, (A-\lambda I)^{d-2} v, \dots, (A-\lambda I) v, v \}$$

$W = \mathrm{Span}\{v\}$ is a L_A-invariant subspace

$$[L_A|_W]_\gamma = \Big([A(A-\lambda I)^{d-1}v]_\gamma \quad [A(A-\lambda I)^{d-2}v]_\gamma \quad \cdots \quad [A(A-\lambda I)v]_\gamma \quad [Av]_\gamma \Big)$$

$$[A(A-\lambda I)^{d-1}v]_\gamma = [\lambda(A-\lambda I)^{d-1}v]_\gamma = \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$[A(A-\lambda I)^{d-2}v]_\gamma = [(A-\lambda I)^{d-1}v + \lambda(A-\lambda I)^{d-2}v]_\gamma = \begin{pmatrix} 1 \\ \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$[A(A-\lambda I)v]_\gamma = \begin{bmatrix} (A-\lambda I)^2 v \\ + \\ \lambda(A-\lambda I)v \end{bmatrix}_\gamma = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \lambda \\ 0 \end{pmatrix}$$

$$[Av]_\gamma = \begin{bmatrix} (A-\lambda I)v \\ + \\ \lambda v \end{bmatrix}_\gamma = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \quad d\times d \text{ matrix.}$$

Ex: $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$

$$-(3-t)+1 \qquad (4-t)\cdot(3-t)-2 = t^2-7t+10 = (t-2)(t-5)$$

$$f(t) = \det(A-tI) = \begin{vmatrix} 3-t & 1 & -2 \\ -1 & -t & 5 \\ -1 & -1 & 4-t \end{vmatrix} = \begin{vmatrix} 0 & t-2 & (t-2)(t-5) \\ 0 & 1-t & 1+t \\ -1 & -1 & 4-t \end{vmatrix}$$

$$= (-1)^{3+1}\cdot(-1)\cdot \begin{vmatrix} t-2 & (t-2)(t-5) \\ 1-t & 1+t \end{vmatrix} = -(t-2)\cdot \begin{vmatrix} 1 & t-5 \\ 1-t & 1+t \end{vmatrix}$$

$$= -(t-2)\cdot[(1+t)-(t-5)\cdot(1-t)] = -(t-2)\cdot[(1+t)+(t^2-6t+5)]$$

$$= -(t-2)\cdot[t^2-5t+6] = -(t-2)^2(t-3). \Rightarrow \lambda=2,\ \mathrm{mult}(2)=2$$

$$\lambda = 3,\ \mathrm{mult}(3)=1.$$

• $\lambda = 2$: $A - 2I = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ $v_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ spans E_2, $\dim E_2 = 1 < 2 = \text{mult}(2)$

$$\Rightarrow \quad \begin{array}{c} 0 \\ \uparrow \\ \bullet\ v_1 \\ \uparrow \\ \bullet\ v_2 \end{array} \quad \Longleftrightarrow \quad \text{Jordan block of size } 2 \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Find v_2: $(A - 2I) v_2 = v_1$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ -1 & -2 & 5 & 3 \\ -1 & -1 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

• $\lambda = 3$: $A - 3I = \begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \quad v_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \text{ spans } E_3 \quad \Longleftrightarrow \quad \begin{array}{c} 0 \\ \uparrow \\ \bullet\ v_3 \end{array}$$

$\Rightarrow$ $Q = (v_1 \ v_2 \ v_3) = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ satisfies

$$Q^{-1} \cdot A \cdot Q = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Ex:

$$\begin{pmatrix} \boxed{\begin{matrix}2&1\\0&2\end{matrix}} & & & \\ & \boxed{2} & & \\ & & \boxed{\begin{matrix}3&1\\0&3\end{matrix}} & \\ & & & \boxed{\begin{matrix}0&1\\0&0\end{matrix}}\end{pmatrix}$$

$$\begin{array}{lccccl} & 0 & 0 & 0 & 0 & \\ & \uparrow A-2I & \uparrow A-2I & \uparrow A-3I & \uparrow A-0\cdot I & \\ \text{Eigenvectors} \to & \bullet & \bullet & \bullet & \bullet & \\ & \uparrow A-2I & & \uparrow A-3I & \uparrow A-0I & \\ & \bullet & & \bullet & \bullet & \leftarrow \text{generalized eigenvectors} \\ \text{cycle of length} & 2 & 1 & 2 & 2 & \end{array}$$

$$(A-\lambda I)^2 v = 0.$$