$T: V \to V$ linear transformation. $\dim V = n$.

β: a basis for V. $A = [T]_\beta$ = matrix representation w.r.t. β

T is diagonalizable $\iff$ $\exists Q$ invertible s.t. $Q^{-1} A Q = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$\iff \exists Q$ invertible s.t. $A = Q \cdot D \cdot Q^{-1}$

$$\Rightarrow \quad A^d = Q \cdot D \cancel{Q^{-1} \cdot Q} \cdot D \cancel{Q^{-1}} \cdots \cancel{Q} D Q^{-1} = Q \cdot D^d Q^{-1}$$

$$\| $$

$$Q \cdot \begin{pmatrix} \lambda_1^d & & & 0 \\ & \lambda_2^d & & \\ & & \ddots & \\ 0 & & & \lambda_n^d \end{pmatrix} Q^{-1}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$

- Check whether A is diagonalizable:

Characteristic polynomial: $f(t) = \begin{vmatrix} 1-t & 2 \\ 5 & 4-t \end{vmatrix} = t^2 - 5t + 4 - 10 = \underset{\substack{\| \\ (t-6)\cdot(t+1)}}{t^2 - 5t - 6} = 0$

$\Rightarrow$ eigenvalues: $\lambda = -1, 6$ $\Rightarrow$ diagonalizable (2 distinct eigenvalues)

- Find eigenvectors:

$\lambda = -1$: $\begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = 6$: $\begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix} \to \begin{pmatrix} 5 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

$\Rightarrow Q = \begin{pmatrix} -1 & 2 \\ 1 & 5 \end{pmatrix}$ satisfies $Q^{-1} A Q = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix} = D$

$Q^{-1} = \frac{1}{-7} \cdot \begin{pmatrix} 5 & -2 \\ -1 & -1 \end{pmatrix} = \frac{1}{7} \cdot \begin{pmatrix} -5 & 2 \\ 1 & 1 \end{pmatrix}$. $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$\Rightarrow \quad A = Q \cdot D \cdot Q^{-1} = Q \cdot \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix} Q^{-1}$$

$$\Rightarrow \quad A^d = Q \cdot \begin{pmatrix} (-1)^d & 0 \\ 0 & 6^d \end{pmatrix} Q^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} (-1)^d & 0 \\ 0 & 6^d \end{pmatrix} \cdot \frac{1}{7} \begin{pmatrix} -5 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{d+1} & 2 \cdot 6^d \\ (-1)^d & 5 \cdot 6^d \end{pmatrix} \cdot \frac{1}{7} \begin{pmatrix} -5 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{7} \cdot \begin{pmatrix} (-1)^d \cdot 5 + 2 \cdot 6^d & 2 \cdot (-1)^{d+1} + 2 \cdot 6^d \\ (-1)^{d+1} \cdot 5 + 5 \cdot 6^d & 2 \cdot (-1)^d + 5 \cdot 6^d \end{pmatrix}$$

For example: $d = 2$: $A^2 = \frac{1}{7} \cdot \begin{pmatrix} 5 + 72 & -2 + 72 \\ -5 + 5 \cdot 36 & 2 + 5 \cdot 36 \end{pmatrix} = \begin{pmatrix} 11 & 10 \\ 25 & 26 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 11 & 10 \\ 25 & 26 \end{pmatrix} \quad \checkmark$$

- Characteristic polynomial of T = Char. polynomial of $[T]_\beta$ for any basis β

If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, then

$$f(t) = |A - tI| = \begin{vmatrix} a_{11} - t & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - t & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - t \end{vmatrix} = (a_{11} - t) \cdots (a_{nn} - t) + g(t) \quad \deg g \leq n-2$$

$$= (-1)^n \big(t^n - \underbrace{(a_{11} + \cdots + a_{nn})}_{= Tr(A)} t^{n-1}\big) + \tilde{g}(t) \quad \deg \tilde{g} \leq n-2$$

$$= b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0 .$$

coefficients: $b_n = (-1)^n$, $b_{n-1} = (-1)^{n-1} \cdot Tr(A)$, $b_0 = f(0) = \det(A)$.

Thm (Cayley-Hamilton Thm). Let $T: V \to V$ be a linear transformation and $f(t) = b_n \cdot t^n + b_{n-1} \cdot t^{n-1} + \cdots + b_1 \cdot t + b_0$ be its characteristic polynomial.

Then
$$0 = f(T) = b_n \cdot \underset{\underbrace{T \circ \cdots \circ T}_{n \text{ times}} \text{ composition.}}{\overset{}{T^n}} + b_{n-1} \cdot T^{n-1} + \cdots + b_1 \cdot T + b_0 \cdot Id.$$

Thm (matrix version) A: $n \times n$ matrix. $f(t) = \det(A - tI)$
$$= b_n \cdot t^n + b_{n-1} \cdot t^{n-1} + \cdots + b_1 t + b_0.$$

Then
$$f(A) = b_n \cdot A^n + b_{n-1} \cdot A^{n-1} + \cdots + b_1 A + b_0 I_n = 0.$$

- If A is diagonalizable, Then $A = Q \cdot D Q^{-1}$ $D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$$\Rightarrow f(A) = b_n \cdot Q \cdot D^n Q^{-1} + b_{n-1} Q \cdot D^{n-1} Q^{-1} + \cdots + b_1 Q \cdot D \cdot Q^{-1} + b_0 \cdot Q \cdot I \cdot Q^{-1}$$
$$= Q \cdot (b_n D^n + b_{n-1} \cdot D^{n-1} + \cdots + b_1 D + b_0 \cdot I) \cdot Q^{-1}$$
$$= Q \begin{pmatrix} b_n \lambda_1^n + b_{n-1} \lambda_1^{n-1} + \cdots + b_1 \lambda_1 + b_0 & & 0 \\ & \ddots & \\ 0 & & b_n \lambda_n^n + b_{n-1} \lambda_n^{n-1} + \cdots + b_1 \lambda_n + b_0 \end{pmatrix} \cdot Q^{-1}$$

$$= Q \cdot \begin{pmatrix} f(\lambda_1) & & & 0 \\ & f(\lambda_2) & & \\ & & \ddots & \\ 0 & & & f(\lambda_n) \end{pmatrix} Q^{-1} = Q \cdot 0 \cdot Q^{-1} = 0.$$

· Proof of Cayley-Hamilton Thm:

It suffices to prove $f(T)\,v=0$ for any $v\in V$.

Consider the T-cyclic subspace generated by v: (we assume $v\neq 0$)

$$W=\mathrm{Span}\{v, Tv, T^2v, \cdots,\}$$

Let m be the largest integer such that $\{v, Tv, \cdots, T^{m-1}v\}$ is linearly independent. Then $\beta'=\{v, Tv, \cdots, T^{m-1}v\}$ is a basis for W.

$\Rightarrow \exists\, c_0, \cdots, c_{m-1}\in \underset{\substack{\uparrow\\ \mathbb{R}\text{ or }\mathbb{C}}}{F}$, s.t. $T^m v = c_0 v + c_1 \cdot Tv + \cdots + c_{m-1}\cdot T^{m-1}v$.

On the other hand, β' can be extended to a basis

$$\beta=\{v, Tv, \cdots, T^{m-1}v, u_1, \cdots, u_k\} \text{ for } V.$$

Calculate the matrix representation:

$$[T]_\beta = \left([T(v)]_\beta\ [T(Tv)]_\beta \cdots [T(T^{m-1}v)]_\beta\ [T(u_1)]_\beta \cdots [T(u_k)]_\beta\right)$$

$$=\left(\begin{array}{ccc|ccc} 0 & 0 & c_0 & * & \cdots & * \\ 1 & 0 & c_1 & & & \\ 0 & 1 & c_2 & & & \\ \vdots & 0 & \vdots & & & \\ \vdots & \vdots & c_{m-1} & & & \\ \hline \vdots & \vdots & 0 & & & \\ 0 & 0 & \vdots & * & \cdots & * \\ & & 0 & & & \end{array}\right)$$

$$=\left(\begin{array}{c|c} \begin{matrix} 0 & 0 & \cdots & c_0 \\ 1 & 0 & \cdots & c_1 \\ 0 & 1 & & c_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots 1 & c_{m-1} \end{matrix} & B \\ \hline O & C \end{array}\right) = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

$$\left|[T]_\beta - tI\right| = \left|\begin{array}{c|c} A - tI_m & B \\ \hline O & C - tI_k \end{array}\right| = \det(A-tI)\cdot\det(C-tI).$$

$$\det(A-tI) = \begin{vmatrix} -t & 0 & \cdots & \cdots & c_0 \\ 1 & -t & & O & c_1 \\ 0 & 1 & \ddots & & \vdots \\ & & \ddots & \ddots & c_{m-2} \\ O & & & 1 & c_{m-1}-t \end{vmatrix}$$

$$= (-t)\cdot \begin{vmatrix} -t & \cdots & \cdots & c_1 \\ 1 & -t & O & c_2 \\ & \ddots & \ddots & c_{m-2} \\ O & & 1 & c_{m-1}-t \end{vmatrix}_{(m-1)\times(m-1)} + \overset{(-1)^{1+m}}{c_0}\cdot \underbrace{\begin{vmatrix} 1 & -t & & * \\ & \ddots & \ddots & c_{m-1}-t \\ O & & & 1 \end{vmatrix}}_{=1}$$

$$= (-t)\cdot\left[(-t)\cdot\begin{vmatrix} -t & \cdots & & c_2 \\ 1 & \ddots & & c_3 \\ & & \ddots & \vdots \\ & & 1 & c_{m-1}-t \end{vmatrix} + (-1)^{1+(m-1)}\cdot c_1\right] + (-1)^{1+m} c_0$$

$$= \cdots = (-t)^{m-2}\cdot\begin{vmatrix} -t & c_{m-2} \\ 1 & c_{m-1}-t \end{vmatrix} + (-1)^{1+m}\cdot\left(c_{m-3}\cdot t^{m-3} + \cdots + c_1\cdot t + c_0\right)$$

$$= (-t)^{m-2}\cdot\left((-t)c_{m-1} + t^2 - c_{m-2}\right) + (-1)^{1+m}\cdot\left(c_{m-3}\cdot t^{m-3} + \cdots + c_1 t + c_0\right)$$

$$= (-1)^{m+1}\cdot\left(-t^m + c_{m-1}\cdot t^{m-1} + c_{m-2}\cdot t^{m-2} + c_{m-3}\cdot t^{m-3} + \cdots + c_1 t + c_0\right)$$

$$f(t) = \det(C - tI_k) \cdot \det(C_1 - tI_m)$$

$$= q(t) \cdot g(t).$$

$$g(T)\cdot v = (-1)^{m+1} \cdot \left[-T^m v + c_{m-1} \cdot T^{m-1} v + c_{m-2} \cdot T^{m-2} v + \cdots + c_1 T v + c_0 v \right)$$

$$= 0$$

$$\Rightarrow f(T)\, v = q(T) \circ g(T)\, v = q(T)\,(0) = 0.$$

This holds for any $v \in V$. $\Rightarrow f(T) = 0$ ∎