$T: V \to V$ is diagonalizable

$\Updownarrow$ def

$\exists$ a basis $\beta = \{v_1, v_2, \cdots, v_n\}$ for V s.t. $\beta(v_i) = \lambda_i v_i$, $\lambda_i \in \underset{\mathbb{R} \text{ or } \mathbb{C}.}{\underset{\uparrow}{F}}$

$\Updownarrow$

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Let γ be a fixed basis for V, $\quad A = [T]_\gamma$

T is diagonalizable $\iff$ $\exists$ invertible matrix Q s.t. $Q^{-1} \cdot A \cdot Q$ is diagonal.

$\iff$ $L_A : \mathbb{R}^n \to \mathbb{R}^n$ is diagonalizable.

Def: characteristic polynomial of $T: V \to V$

$$f(t) = \det\left([T]_\gamma - t \cdot I\right) \text{ for any basis } \gamma.$$

Lem: $f(t)$ does not depend on the choice of basis γ.

Proof: If β is a different basis, then

$$B = [T]_\beta = [T]_\beta^\beta = [T]_\gamma^\beta [T]_\gamma^\gamma \cdot [T]_\beta^\gamma = \overset{Q^{-1} A \cdot Q}{\overset{\parallel}{Q^{-1} [T]_\gamma \cdot Q}}$$

$$B - tI = Q^{-1} A Q - t \cdot Q^{-1} \cdot I \cdot Q = Q^{-1} (A - tI) \cdot Q$$

$$\Rightarrow \det(B - tI) = \det(Q)^{-1} \cdot \det(A - tI) \cdot \det(Q) = \det(A - tI). \quad \blacksquare$$

Prop: λ is an eigenvalue of $T \Leftrightarrow f(\lambda)=0$ $\quad (T-\lambda I)v=0$

Proof: λ is an eigenvalue $\Leftrightarrow \exists\, v\neq 0$, s.t. $Tv=\lambda v$

$\Leftrightarrow$ nullity $(T-\lambda I)>0 \Leftrightarrow$ rank $(T-\lambda I)=n-$ nullity $(T-\lambda I)<n$

$\Leftrightarrow \det(T-\lambda I)=0 \Leftrightarrow f(\lambda)=0$ ∎

Def: $\cdot$ λ is an eigenvalue $\Rightarrow f(t)=(t-\lambda)^{m_\lambda}g(t)$ with $g(\lambda)\neq 0$, m_λ an integer

m_λ is called the multiplicity of the eigenvalue λ.
$\parallel$
$\text{mult}(\lambda)$

$\cdot$ eigenspace: $E_\lambda = N(T-\lambda I)=\{v\in V: T(v)=\lambda\cdot v\}$

Prop: $\dim E_\lambda \leq \text{mult}(\lambda)$.

Proof: Choose a basis β' for E_λ. Extend β' to a basis β for V

Then
$$[T]_\beta = \left(\begin{array}{c|c} \begin{matrix}\lambda & & 0\\ & \ddots & \\ 0 & & \lambda\end{matrix} & B \\ \hline O & C\end{array}\right)\begin{matrix}\}k\\ \}n-k\end{matrix}$$

$$\Rightarrow \det([T]_\beta - tI) = \left|\begin{array}{c|c} \lambda I - tI & B \\ \hline 0 & C-t\cdot I_{n-k}\end{array}\right| = (\lambda-t)^k\cdot\det(C-tI_{n-k})$$

$\Rightarrow k=\dim E_\lambda \leq \text{mult}(\lambda)$. ∎

Thm: T is diagonalizable if and only if the following two conditions are satisfied.

(i) The characteristic polynomial of T splits. i.e.

$$f(t) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$$

(ii) $\dim E_\lambda = \mathrm{mult}(\lambda)$ for each eigenvalue λ.

Proof:

Assume $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues.

Let β_i be a basis for E_{λ_i}.

Then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a linearly independent subset.

T is diagonalizable $\Longleftrightarrow$ β is a basis for V

$$\Longleftrightarrow \sum_{i=1}^{k} \dim E_{\lambda_i} = n = \sum_{i=1}^{k} \mathrm{mult}(\lambda_i)$$

$$\Longleftrightarrow \dim E_{\lambda_i} = \mathrm{mult}(\lambda_i) \text{ for any } i = 1, \cdots, k$$

$(\dim E_{\lambda_i} \leq \mathrm{mult}(\lambda_i))$ and $\sum_{i=1}^{k} \mathrm{mult}(\lambda_i) = n$

$\Longleftrightarrow$ The characteristic polynomial of T splits and $\dim E_{\lambda_i} = \mathrm{mult}(\lambda_i)$ for each $i = 1, \cdots, k$.

Prop: S_i linear independent subset of E_{λ_i}, with $\{\lambda_i ; i=1,\dots,k\}$ distinct. $\Rightarrow$ $S_1 \cup \cdots \cup S_k$ is linearly indep.

Proof: Let $S_i = \{ v_1^{(i)}, \dots, v_{d_i}^{(i)} \}$, $i=1,\dots,k$.

Assume there is a linear relation:

$$\left(a_{11} v_1^{(1)} + \cdots + a_{1d_1} v_{d_1}^{(1)}\right) + \left(a_{21} v_1^{(2)} + \cdots + a_{2d_2} v_{d_2}^{(2)}\right) + \cdots$$
$$\cdots + \left(a_{k1} v_1^{(k)} + \cdots + a_{kd_k} v_{d_k}^{(k)}\right) = 0.$$
$$\|$$
$$u_1 + u_2 + \cdots + u_k = 0 \quad \text{with } u_i \in E_{\lambda_i}$$

Apply T: $Tu_1 + Tu_2 + \cdots + Tu_k = 0$
$$\|$$
$$\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k = 0$$

$\Rightarrow$ $(\lambda_2 - \lambda_1) u_2 + \cdots + (\lambda_k - \lambda_1) u_k = 0$. $\quad \lambda_i - \lambda_1 \neq 0, i \neq 1$

Use induction: $k=1$ case true because S_1 is linearly ind.

Assume $(k-1)$th case is true $\Rightarrow u_2 = \cdots = u_k = 0$

$\Rightarrow u_1 = 0 \Rightarrow a_{jm}^{(i)} = 0 \ \forall i, j, m$

∎