$T: V \to V$ a linear transformation.

Def: T is called diagonalizable if there exists a basis $\beta = \{v_1, v_2, \cdots, v_n\}$ s.t. for any $i=1,\cdots,n$, $Tv_i = \lambda_i v_i$ for some $\lambda_i \in F$

$F \leftarrow$ field that can be $\mathbb{R}$ or $\mathbb{C}$

$\mathbb{C} \leftarrow$ complex numbers.

$$\Rightarrow [T]_\beta = \left([T(v_1)]_\beta \ [T(v_2)]_\beta \cdots [T(v_n)]_\beta \right)$$
$$= \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

- Let γ be any fixed basis for V. Then
$$[T]_\beta = [T]_\beta^\beta = [Id_V \circ T \circ Id_V]_\beta^\beta = [Id_V]_\gamma^\beta [T]_\gamma^\gamma \cdot [Id_V]_\beta^\gamma$$
Let $P = [Id_V]_\beta^\gamma$ be the matrix of coordinate change. Then:
$$[T]_\beta = P^{-1} \cdot [T]_\gamma \cdot P. \leftarrow \text{conjugation of } [T]_\gamma$$

So T is diagonalizable $\Leftrightarrow$ $[T]_\gamma$ can be transformed to a diagonal matrix by an invertible matrix P as above.

- Set $A = [T]_\gamma$, $P = (v_1 v_2 \cdots v_n)$ with $v_i \in \mathbb{R}^n$.

Then $P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{pmatrix} \Leftrightarrow A \cdot P = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{pmatrix} P$

$$A \cdot P = A(v_1\, v_2 \cdots v_n) = (Av_1\, Av_2 \cdots Av_n)$$
$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{pmatrix} P = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{pmatrix} (v_1 \cdots v_n) = (\lambda_1 v_1\ \lambda_2 v_2 \cdots \lambda_n v_n).$$

A is diagonalizable $\Leftrightarrow \exists P$ invertible s.t. $P^{-1}AP$ is diagonal.

$\Leftrightarrow \exists$ a basis $\{v_1, \cdots, v_n\}$ for $\mathbb{R}^n$ s.t.

$Av_i = \lambda_i v_i$, $i=1, \cdots, n$.

$\Leftrightarrow L_A: \mathbb{R}^n \to \mathbb{R}^n$ is diagonalizable.

· Def: · $v \neq 0 \in V$ is an eigenvector for $T: V \to V$ if

$Tv = \lambda v$ for some $\lambda \in F$.

· $v \neq 0 \in \mathbb{R}^n$ is an eigenvector for $A \in M_{n\times n}$ if

$Av = \lambda v$ for some $\lambda \in F$.

· How to find eigenvalues:

λ is an eigenvalue for $A \Leftrightarrow \exists v \neq 0 \in \mathbb{R}^n$ s.t. $Av = \lambda v$ $\Updownarrow$ $(A-\lambda I)v = 0$.

$\Leftrightarrow N(A-\lambda I) \neq \{0\}$ i.e. nullity$(A-\lambda I) > 0$.

$\Leftrightarrow$ rank$(A-\lambda I) = n -$ nullity$(A-\lambda I) < n$

$\Leftrightarrow \det(A-\lambda I) = 0$.

Def: $f(t) = \det(A-tI)$ is called the characteristic polynomial for A.

So: λ is an eigenvalue for $A \Leftrightarrow \lambda$ is a root to the characteristic polynomial.

$E_\lambda = \{$eigenvectors with eigenvalue $\lambda\} = \underline{N(A-\lambda I)}$.

$\dim N(A-\lambda I) = \#$ free variables in rref$(A-\lambda I)$.

__Example:__

$$A=\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$$

1. Find eigenvalues: First calculate the characteristic polynomial:

$$f(t)=\det(A-tI)=\begin{vmatrix} -t & -2 & -3 \\ -1 & 1-t & -1 \\ 2 & 2 & 5-t \end{vmatrix}=\begin{vmatrix} 0 & t^2-t-2 & t-3 \\ -1 & 1-t & -1 \\ 0 & 4-2t & 3-t \end{vmatrix}$$

$$\|$$

$$+(t-2)\cdot(t-3)\cdot[-t-1+2]=+(t-2)\cdot(t-3)\cdot\begin{vmatrix} t+1 & 1 \\ -2 & -1 \end{vmatrix}=-\cdot(-1)\cdot\begin{vmatrix} (t-2)(t+1) & 1 \\ 2(2-t) & -1 \end{vmatrix}\cdot(t-3)$$

$$\|$$

$$(t-2)\cdot(t-3)\cdot(t-1)=0$$

$$\Rightarrow \lambda=1,2,3.$$

2. Find eigenvectors:

$\lambda=1$: $$A-\lambda I=\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix}\rightarrow\begin{pmatrix} 0 & -2 & -2 \\ -1 & 0 & -1 \\ 0 & 2 & 2 \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_1=\text{Span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$\lambda=2$: $$A-\lambda I=\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix}\rightarrow\begin{pmatrix} 0 & 0 & -1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_2=\text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}.$$

$\lambda=3$: $$A-\lambda I=\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix}\rightarrow\begin{pmatrix} 0 & 4 & 0 \\ -1 & -2 & -1 \\ 0 & -2 & 0 \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_3=\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

$\Rightarrow$ A is diagonalizable.

We get 3 eigenvectors associated to 3 different eigenvalues. They are linearly independent and form a basis for $\mathbb{R}^3$ because of the following Theorem.

<u>Thm</u>: Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigenvalues.

Assume

$S_1 = \{v_1^{(1)}, \cdots, v_{d_1}^{(1)}\}$ is a linearly independent subset of $E\lambda_1$

$S_2 = \{v_1^{(2)}, \cdots, v_{d_2}^{(2)}\}$ is a linearly independent subset of $E\lambda_2$

$\cdots$

$S_k = \{v_1^{(k)}, \cdots, v_{d_k}^{(k)}\}$ is a linearly independent subset of $E\lambda_k$

Then $S_1 \cup S_2 \cup \cdots \cup S_k$ is linearly independent.

Eg:
$$A=\begin{pmatrix}0&-2&-3\\-1&1&-1\\1&3&4\end{pmatrix}$$

1. $$|A-tI|=\begin{vmatrix}-t&-2&-3\\-1&1-t&-1\\1&3&4-t\end{vmatrix}=\begin{vmatrix}0&t^2-t-2&t-3\\-1&1-t&-1\\0&4-t&3-t\end{vmatrix}=-(-1)(t-3)\cdot\begin{vmatrix}t^2-t-2&1\\0&0\\4-t&-1\end{vmatrix}$$

$$=(t-3)\cdot\left(-(t^2-t-2)-(4-t)\right)=-(t-3)\cdot(t^2-2t+2).$$

$\Rightarrow \lambda=3.$

2. Find E_3:
$$A-3I=\begin{pmatrix}-3&-2&-3\\-1&-2&-1\\1&3&1\end{pmatrix}\to\begin{pmatrix}0&4&0\\-1&-2&-1\\0&1&0\end{pmatrix}\to\begin{pmatrix}1&0&-1\\0&1&0\\0&0&0\end{pmatrix}$$

$\Rightarrow E_3=\text{Span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}.$

over $\mathbb{R}$. A is not diagonalizable.

---

over $\mathbb{C}$, there are two more eigenvalues:

$$\lambda^2-2\lambda+2=(\lambda-1)^2+1=0\Rightarrow\lambda-1=\sqrt{-1}=\pm i\Rightarrow\lambda=1+i,\ 1-i$$

Find E_{1+i}:
$$A-(1+i)I\rightsquigarrow\begin{pmatrix}0&i-3&-2+i\\-1&-i&-1\\0&3-i&2-i\end{pmatrix}\to\begin{pmatrix}1&i&1\\0&1&\frac{7-i}{10}\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&\frac{7-11i}{10}\\0&1&\frac{7-i}{10}\\0&0&0\end{pmatrix}$$

$(1+i)^2-(1+i)-2=2i-3-i=i-3$, $\quad\frac{2-i}{3-i}=\frac{(2-i)\cdot(3+i)}{(3-i)(3+i)}=\frac{7-i}{10}$

$$\Rightarrow \quad E_{1+i} = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix} -\frac{7-11i}{10} \\ -\frac{7-i}{10} \\ 1 \end{pmatrix}}_{= v} \right\}$$

$$\Rightarrow \quad E_{1-i} = \operatorname{Span}\{\bar{v}\}.$$

$$\boxed{\begin{aligned} &Av = \lambda v \\ &\Downarrow \\ &A\bar{v} = \bar{\lambda}\bar{v} \end{aligned}}$$

eigenvectors over $\mathbb{C}$.

$$\Rightarrow \quad \beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v, \bar{v} \right\} \text{ form a basis for } \mathbb{C}^3$$

$\Rightarrow$ A is diagonalizable over $\mathbb{C}$.