$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad n\times n \text{ matrix.}$$

$$|A| = \det(A) = a_{11}\cdot\det(A_{11}) - a_{12}\cdot\det(A_{12}) + \cdots + a_{1j}\cdot(-1)^{1+j}\det(A_{1j}) + \cdots + a_{1n}\cdot(-1)^{1+n}\cdot\det(A_{1n})$$

Property 1: $$\det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ cu+v \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix} = c\cdot\det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ u \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix} + \det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ v \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix}.$$

$$\Rightarrow \det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ \sum_j a_{kj}\cdot e_j \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix} = \sum_{j=1}^{n} a_{kj}\cdot\det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ e_j \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix}$$

Lemma: $$\det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ e_j \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix} = (-1)^{k+j}\det(A_{kj}).$$ (Proof by induction)

$\Rightarrow$ expansion along any row: $\det(A) = \sum_{j=1}^{n} a_{kj}\cdot(-1)^{k+j}\det(A_{kj})$.

Cor: If A has two identical rows, then $\det(A)=0$.

Pf: Use induction. $n=2$: $\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} = a_{11}a_{12} - a_{12}a_{11} = 0$.

$n\geq 3$: expand along the row that is different from the identical rows and use induction.

Cor: $$\det\begin{pmatrix} \vdots \\ r_j \\ \vdots \\ r_i \\ \vdots \end{pmatrix} = -\det\begin{pmatrix} \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \end{pmatrix}$$ switch 2 rows $\rightsquigarrow$ determinant change sign.
(elementary row operation of type I).

Cor: Elementary row operation of type 3 does not change the determinant.
add a multiple of one row to another row.

Cor: $\det(E \cdot A) = \det(E) \cdot \det(A)$ for any elementary matrix E and square matrix A.

Pf: $E = E^{(i) \leftrightarrow (j)}$: $\det(EA) = -\det(A) = \det(E) \cdot \det(A)$

$E = E^{c \cdot (i)}$: $\det(EA) = c \cdot \det(A) = \det(E) \cdot \det(A)$

$E = E^{c \cdot (i) + (j)}$: $\det(EA) = \det(A) = \det(E) \cdot \det(A)$ ∎

Thm: $\operatorname{rank}(A) < n \Leftrightarrow \det(A) = 0$

Pf: $\operatorname{rank}(A) < n \Rightarrow$ rows of A are linearly dependent

$\Downarrow$

$\det(A) = 0 \Leftarrow$ one row is a linear combination of other rows

$\operatorname{rank}(A) = n \Leftrightarrow$ reduced echelon form of A $\operatorname{rref}(A) = I_n$

$\Downarrow$

$A = E_1 \cdots E_p$ is a product of elementary matrices

$\Downarrow$

$\det(A) = \det(E_1) \cdots \det(E_p) \neq 0$. ∎

Summarize: $\operatorname{rank}(A) = n \Leftrightarrow \det(A) \neq 0$

$\Updownarrow$ $\Updownarrow$

columns of A are linearly independent

$\operatorname{rref}(A) = I_n \Leftrightarrow A = E_1 \cdots E_p$ is a product of elementary matrices.

Thm: $\det(AB) = \det(A)\cdot\det(B)$.

Case 1: $\operatorname{rank}(A) < n \iff \det(A) = 0$.

$$\operatorname{rank}(AB) = \dim R(L_{AB}) \leq \dim R(L_A) = \operatorname{rank}(A) < n$$

$$L_{AB} = L_A \circ L_B \Rightarrow R(L_{AB}) \subseteq R(L_A).$$

$\Rightarrow \det(AB) = 0$.

Case 2: $\operatorname{rank}(A) = n \Rightarrow A = E_1 \cdots E_p$ E_i: elementary matrices.

$$\det(AB) = \det(E_1 \cdots E_p B) = \det(E_1)\cdots\det(E_p)\cdot\det(B)$$
$$= \det(A)\cdot\det(B).$$

Thm: $\det(A^t) = \det(A)$.

Pf: Case 1:

$$\begin{array}{ccc} \operatorname{rank}(A) < n & \iff & \det(A) = 0 \\ \| & & \| \\ \operatorname{rank}(A^t) < n & \iff & \det(A^t) = 0 \end{array}$$

Case 2: $\operatorname{rank}(A) = n \iff A = E_1 \cdots E_p$

$$\Downarrow$$

$$\det(A) = \det(E_1)\cdots\det(E_p)$$

$$\|$$

$$A^t = E_p^t \cdots E_1^t \Rightarrow \det(A^t) = \det(E_1^t)\cdots\det(E_p^t)$$

$$(E^{(i)\leftrightarrow(j)})^t = E^{(i)\leftrightarrow(j)}, \quad (E^{c(i)})^t = E^{c(i)}, \quad (E^{c(i)+(j)})^t = E^{c(j)+(i)}.$$

$$\begin{array}{cc} i & \\ j & \end{array}\begin{pmatrix} \cdots 0 \cdots 1 \cdots \\ \cdots 1 \cdots 0 \cdots \end{pmatrix} \qquad \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \qquad \begin{array}{cc} i & \\ j & \end{array}\begin{pmatrix} \cdots 1 \cdots 0 \cdots \\ \cdots c \cdots 1 \cdots \end{pmatrix}^t = \begin{pmatrix} \cdots 1 \cdots c \cdots \\ \cdots 0 \cdots 1 \cdots \end{pmatrix}$$

Example: Upper triangular matrix

$$\det\begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ & & \ddots & \\ 0 & & & a_{nn}\end{pmatrix} = a_{11}a_{22}\cdots a_{nn}$$

This can be shown by expanding along the 1st. row and use induction.

Example:

$$\begin{vmatrix} 0 & 1 & 3 & -3 \\ 2 & 0 & 0 & 1 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{vmatrix} \overset{1 \leftrightarrow 2}{=\!=} -\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & -3 & -5 & 3 \\ 0 & -4 & 4 & -8 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 16 & -20 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 4 \end{vmatrix} = -2\cdot 1\cdot 4\cdot 4 = -32$$

$$\begin{vmatrix} 0 & 1 & 3 & -3 \\ 2 & 0 & 0 & 1 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{vmatrix} \overset{\boxed{1} \leftrightarrow \boxed{2}}{=\!=} -\begin{vmatrix} 1 & 0 & 3 & -3 \\ 0 & 2 & 0 & 1 \\ -3 & -2 & -5 & 2 \\ -4 & 4 & 4 & -6 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 3 & -3 \\ 0 & 2 & 0 & 1 \\ 0 & -2 & 4 & -7 \\ 0 & 4 & 16 & -18 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 0 & 1 \\ -2 & 4 & -7 \\ 4 & 16 & -18 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 1 \\ 0 & 4 & -6 \\ 0 & 16 & -20 \end{vmatrix} = -2\cdot\begin{vmatrix} 4 & -6 \\ 16 & -20 \end{vmatrix}$$

$$= -2\cdot 2\cdot 4\cdot\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix} = -16\cdot(-10+12) = -32$$

Method:

Use elementary row operations to transform A to simpler matrices, for example upper-triangular matrix, combined with expansion along any row or column.

- Characterization of determinant function.

<u>Thm:</u> Assume a function $\delta: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ satisfies 3 properties:

(1) δ is linear w.r.t. one row when other rows are fixed.

(2) If $A \xrightarrow{① \leftrightarrow ②} B$, then $\delta(B) = -\delta(A)$.

(3) $\delta(I_{n\times n}) = 1$.

Then $\delta = \det$.

Idea of proof: · Prove $\delta(E) = \det(E)$ for any elementary matrix E

- Prove $\delta(AB) = \delta(A)\cdot\delta(B)$ by using similar arguments for det.

- $\operatorname{rank}(A) < n \Rightarrow \delta(A) = 0$.
  $\operatorname{rank}(A) = n \Rightarrow \delta(A) \neq 0$.

- Prove $\delta(A) = \det(A)$:

  <u>Case 1:</u> $\operatorname{rank}(A) < n \Rightarrow \delta(A) = 0 = \det(A)$

  <u>Case 2:</u> $\operatorname{rank}(A) = n \Rightarrow A = E_1 \cdots E_p$

$$\Downarrow$$

$$\delta(A) = \delta(E_1)\cdots\delta(E_p) = \det(E_1)\cdots\det(E_p) = \det(A)$$

∎