$Ax = b \rightsquigarrow$ corresponding homogeneous system $Ax=0$.

$$K = \{\text{solutions to } Ax=b\} = s + \underset{\substack{\| \\ \{\text{solutions to } Ax=0\} \\ \| \\ N(L_A).}}{K_H}$$

To solve the linear system $Ax=b$, need

- Determine whether the system is consistent (i.e. whether there is any solutions)

  consistent $\Longleftrightarrow$ $\operatorname{rank}(A|b) = \operatorname{rank}(A)$.

- Describe the subspace $K_H = N(L_A)$ by finding a basis for it.

To achieve these 2 goals, we use elementary row operation to transform the augmented matrix into its reduced row echelon form.

$$(A|b) \longrightarrow (A'|b')$$

Then the answers to the above questions can be easily answered.

reduced echelon form of $(A|b)$
- decide consistency or not
- find a basis for the column space
- find a basis for the row space

Example from last lecture:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & \frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- $\mathrm{rank}(A|b) = 2 = \mathrm{rank}(A) \Rightarrow$ consistent.
- particular solution: $\begin{pmatrix} \frac{5}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix}$
- solutions to the homogeneous system:

$$\begin{cases} x_1 = -x_2 - \frac{4}{3}x_4 \\ x_3 = \frac{2}{3}x_4 \end{cases}$$

x_2, x_4 are free variables

write down basis for $N(A)$: set $x_2 = 1, x_4 = 0$: $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$x_2 = 0, x_4 = 1$: $\begin{pmatrix} -\frac{4}{3} \\ 0 \\ \frac{2}{3} \\ 1 \end{pmatrix}$

$$N(A) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{3} \\ 0 \\ \frac{2}{3} \\ 1 \end{pmatrix} \right\}$$

- a basis for the column space:

columns corresponding to leading variables.

column space of $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 1 & 2 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$

free variables.

elementary row operations do not change the linear relations between column vectors.

$$A=(v_1\ v_2 \cdots v_n) \rightsquigarrow A'=(v_1'\ v_2' \cdots v_n').$$

$$c_{i_1}v_{i_1}+\cdots+c_{i_k}v_{i_k}=0 \iff c_{i_1}v_{i_1}'+\cdots+c_{i_k}v_{i_k}'=0$$

$$\begin{pmatrix} 1 & & -1 & \\ 1 & v_2 & 2 & v_4 \\ 2 & & 1 & \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

recover v_2 and v_4: $v_2=v_1=\begin{pmatrix}1\\1\\2\end{pmatrix} \longleftarrow v_2'=v_1'$

$v_4=\frac{4}{3}v_1-\frac{2}{3}v_3 \qquad v_4'=\frac{4}{3}v_1'-\frac{2}{3}v_3'$

$$=\frac{4}{3}\begin{pmatrix}1\\1\\2\end{pmatrix}-\frac{2}{3}\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}2\\0\\2\end{pmatrix}$$

reduced echelon form + corresponding column basis vectors

$\xrightarrow{\text{recover}}$ all other column vectors.

- basis for row space

elementary row operations do not change the row space.

(reduced) row echelon form $\rightsquigarrow$ basis for row space of row echelon form

$\rightsquigarrow$ basis for A

Ex. $$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & \frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{row space}=\operatorname{Span}\left\{\begin{matrix}(1,1,-1,2)\\(1,1,2,0)\\(2,2,1,2)\end{matrix}\right\}=\operatorname{Span}\left\{\begin{matrix}(1,1,0,\frac{4}{3})\\(0,0,1,-\frac{2}{3})\end{matrix}\right\} \text{ basis.}$$

- Determinant. $\det(A) = |A|$

Inductive Definition: $\det(a_{11}) = a_{11}$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}$$

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = a_{11}\cdot \underbrace{\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ & \cdots & \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{\det \text{ of } (n-1)\times(n-1)} - a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ & & \cdots & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

$$+ \cdots + (-1)^{1+n}\cdot a_{1n}\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2(n-1)} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} \end{vmatrix}$$

expansion along the 1st. row.

Thm: $$\det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ u + c\cdot v \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix} = \det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ u \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix} + c\cdot\det\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ v \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix}$$

det function is linear for a given row when all other rows are fixed.

Note in general, for $n\times n$ matrices A, B

$$\det(A+B) \neq \det(A) + \det(B)$$

$$\det(c\cdot A) = c^n \cdot \det(A)$$

Pf of Thm is by induction (because each term on the right is already a linear combination by induction)