Linear System of equations

$$x=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\in\mathbb{R}^n,\quad b=\begin{pmatrix}b_1\\b_2\\\vdots\\b_m\end{pmatrix}\in\mathbb{R}^m.$$

$$\begin{cases}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2\\ \quad\cdots\cdots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m\end{cases}\iff Ax=b$$

Coefficient matrix $A=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\ &\cdots&\cdots&\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{pmatrix}$

- homogeneous system: $Ax=0$
- inhomogeneous system: $Ax=b\neq 0 \rightsquigarrow$ corresponding homogeneous system: $Ax=0$.

- For a homogeneous system: $Ax=0$

$$s \text{ is a solution} \iff \underset{\substack{\|\\ L_A(s)}}{As}=0 \iff s\in N(L_A) \qquad L_A:\mathbb{R}^n\to\mathbb{R}^m \quad L_A(v)=A\cdot v.$$

$$\text{So } \overset{\substack{K_H\\ \|}}{\{\text{solutions to } Ax=0\}}=N(L_A) \text{ is a subspace of } \mathbb{R}^n$$

$$\dim\{\text{solutions to } Ax=0\}=\dim N(L_A)=\underset{\substack{\|\\ n-\operatorname{rank}(L_A)}}{\text{nullity}(L_A)}$$

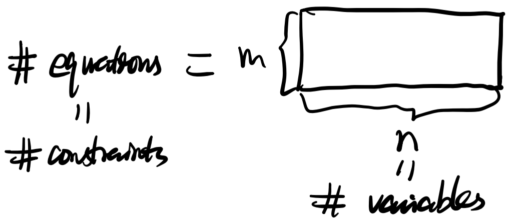
$$\underset{\substack{\|\\ n-\operatorname{rank}(A)\ =}}{}$$

- $\operatorname{rank}(A)=\dim(\underbrace{\text{column space}}_{\substack{\cap\\ \mathbb{R}^m}})\leq n \Rightarrow \dim K_H=\underset{\substack{\vee\\ n-m.}}{n-\operatorname{rank}(L_A)}$

  miracle $\|$

  $\dim(\underbrace{\text{row space}}_{\substack{\cap\\ \mathbb{R}^n}})\leq m$

So Corollary if $m < n$, then $\dim K_H > 0$. So there are always nonzero solutions to the homogeneous linear system if # of equations < # of variables.



- A nonhomogeneous system $Ax = \overset{\overset{0}{\neq}}{b}$ is called

consistent : if there is a solution

inconsistent : if there are no solutions.

$$Ax=b \text{ is consistent} \iff \exists\, s\in\mathbb{R}^n \text{ s.t. } \overset{\overset{L_A(s)}{\shortparallel}}{As} = b \in \mathbb{R}^m$$

$$\Updownarrow$$

$$b\in R(L_A).$$

Thm: $Ax=b$ is consistent if and only if $\operatorname{rank}(A)=\operatorname{rank}(A|b)$

where $$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \ddots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$ is called the augmented matrix.

$$\shortparallel$$

$$(v_1 \; v_2 \; \cdots \; v_m \,|\, b)$$

Pf: $Ax=b$ is consistent $\iff \underline{b\in \left(R(L_A) = \operatorname{span}\{v_1, v_2, \cdots, v_n\}\right)}$

$$\Updownarrow$$

$$\underset{\operatorname{rank}(A|b)}{\underset{\shortparallel}{\dim \operatorname{Span}\{v_1,v_2,\cdots,v_n,b\}}} = \underset{\operatorname{rank}(A).}{\underset{\shortparallel}{\dim\operatorname{Span}\{v_1,v_2,\cdots,v_n\}}} \iff \operatorname{Span}\{v_1,v_2,\cdots,v_n,b\} = \operatorname{span}\{v_1,v_2,\cdots,v_n\}$$

Thm: Assume $s \in \mathbb{R}^n$ is a solution to $Ax=b$.

Then any solution v to $Ax=b$ is equal to $v = s + (v-s)$

with $(v-s)$ being a solution to the corresponding homogeneous system $Ax=0$.

Pf: $\begin{matrix} As = b \\ Av = b \end{matrix} \Rightarrow A(v-s) = Av - As = 0$ ■

So
$$\underset{\substack{\uparrow \\ \{\text{solutions to } Ax=b\}}}{K} = \underset{\substack{\uparrow \\ \text{a particular} \\ \text{solution to} \\ Ax=b}}{s} + \underset{\substack{\uparrow \\ \{\text{solutions to } Ax=0\}}}{K_H} = \{s+u : u \in K_H\}.$$

Example:
$$\begin{cases} x_1 + x_2 - x_3 + 2x_4 = 2 \\ x_1 + x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + x_3 + 2x_4 = 4 \end{cases} \iff \begin{cases} x_1 + x_2 - x_3 + 2x_4 = 2 \\ 3x_3 - 2x_4 = -1 \\ 3x_3 - 2x_4 = 0 \end{cases}$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 4 \end{array}\right) \xrightarrow[①\cdot(-2)+③]{①\cdot(-1)+②} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 3 & -2 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + x_2 - x_3 + 2x_4 = 2 \\ x_3 - \frac{2}{3}x_4 = -\frac{1}{3} \\ \boxed{0 = 1} \end{cases} \quad \text{in consistent}$$

$$\updownarrow$$

$$\xrightarrow[②/3]{②\cdot(-1)+③} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad \underset{\substack{\| \\ 3}}{\operatorname{rank}(A'|b')} \neq \underset{\substack{\| \\ 2}}{\operatorname{rank}(A')}$$

$$\underset{\substack{\| \\ A'}}{} \qquad \underset{\substack{\| \\ b'}}{}$$

elementary row operations transform a linear system into equivalent linear systems

$$\Updownarrow$$

multiplication on the left by elementary row operations does not change the rank of coefficient matrices and the ranks of augmented matrices.

<u>Example:</u> slight change of the last example

$$\begin{cases} x_1+x_2-x_3+2x_4=2 \\ x_1+x_2+2x_3 \qquad =1 \\ 2x_1+2x_2+x_3+2x_4=3 \end{cases} \Longleftrightarrow \begin{cases} x_1+x_2-x_3+2x_4=2 \\ \qquad 3x_3-2x_4=-1 \\ \qquad 3x_3-2x_4=-1 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 4 \end{array}\right) \xrightarrow[①\cdot(-2)+③]{①\cdot(-1)+②} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 3 & -2 & -1 \end{array}\right)$$

$$\begin{cases} x_1+x_2-x_3+2x_4=2 \\ \qquad x_3-\frac{2}{3}x_4=-\frac{1}{3} \\ \qquad\qquad 0=0 \end{cases} \qquad \begin{cases} x_1+x_2+\frac{4}{3}x_4=\frac{5}{3} \\ \qquad x_3-\frac{2}{3}x_4=-\frac{1}{3} \\ \qquad\qquad 0=0 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\xrightarrow[②/3]{②\cdot(-1)+③} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{②\cdot(-1)+①} \left(\begin{array}{cccc|c} 1 & 1 & 0 & \frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

leading variables

free variables

$\rightarrow$ all solutions
$$\begin{cases} x_1=\frac{5}{3}-x_2-\frac{4}{3}x_4 \\ x_3=-\frac{1}{3}+\frac{2}{3}x_4 \end{cases}$$

$$\Rightarrow \text{Any solution} \quad \underset{\substack{\nearrow \\ \text{general solution to} \\ Ax=b}}{\begin{pmatrix} \frac{5}{3}-x_2-\frac{4}{3}x_4 \\ x_2 \\ -\frac{1}{3}+\frac{2}{3}x_4 \\ x_4 \end{pmatrix}} = \underset{\substack{\uparrow \\ \text{particular} \\ \text{solution to} \\ Ax=b}}{\begin{pmatrix} \frac{5}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix}} + \underbrace{x_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -\frac{4}{3} \\ 0 \\ \frac{2}{3} \\ 1 \end{pmatrix}}_{\substack{\text{general solution to} \\ Ax=0}}$$

$$K = s + K_H$$

$$K_H = \{\text{solutions to } Ax=b\} = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{3} \\ 0 \\ \frac{2}{3} \\ 1 \end{pmatrix} \right\} = N(L_A).$$

Gauss ellimination ⇝ reduced echelon form for $(A|b)$

⇝ solutions to inhomogeneous systems

⇝ basis for $N(L_A)$ ← $\dim N(L_A) = \#$ free variables

⇝ $\boxed{\text{basis for } R(L_A)}$ ← $\dim R(L_A) = \#$ leading variables

left multiplication by elementary matrices does not change the linear relation between column vectors ⇒ column vectors corresponding to leading variables form a basis for the column space:

Example above: $R(L_A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\}$

$$= \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$