$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad m \times n \quad \text{matrix}$$

Def: Elementary row operations:

Type 1: interchange 2 rows. $(i) \leftrightarrow (j)$: interchange ith and jth row.

Type 2: multiply a row by a nonzero scalar $(i) \to c \cdot (i)$

Type 3: add a multiple of one row to another row. $(j) \to c \cdot (i) + (j)$.

- An elementary matrix E is a matrix obtained from the ~~identity~~ matrix by applying an elementary operation:

$$I_{m\times m} \xrightarrow{(i)\leftrightarrow(j)} E^{(i)\leftrightarrow(j)} = \begin{array}{c} \\ i \\ j \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} & & i & & j & \end{array} \\ \begin{pmatrix} 1 & \ddots & & 0 & & 0 \\ & & 0 & \cdots & 1 & \\ & 0 & \vdots & 1 \ddots 1 & \vdots & 0 \\ & & 1 & \cdots & 0 & \\ 0 & & 0 & & \ddots & 1 \end{pmatrix} \end{array}$$

$$I_{m\times m} \xrightarrow{c\cdot(i)} E^{c\cdot(i)}$$

$$I_{m\times m} \xrightarrow{c(i)+(j)} E^{c\cdot(i)+(j)}$$

Thm:

$$A \xrightarrow{(i)\leftrightarrow(j)} E^{(i)\leftrightarrow(j)} \cdot A$$

$$A \xrightarrow{c\cdot(i)} E^{c\cdot(i)} \cdot A$$

$$A \xrightarrow{c(i)+(j)} E^{c(i)+(j)} \cdot A$$

For example,

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & 1 \ddots 1 & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & 1 \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \\ r_m \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_i \\ \vdots \\ r_m \end{pmatrix}$$

Thm: An elementary matrix is invertible.

Pf:
$$E^{(i)\leftrightarrow(j)} \cdot E^{(i)\leftrightarrow(j)} = I_{m\times m}$$
$$E^{c\cdot(i)} \cdot E^{c^{-1}\cdot(i)} = I_{m\times m}$$
$$E^{(-c)(i)+(j)} \circ E^{c(i)+(j)} = \left(E^{c(i)+(j)}\right)^{(-c)(i)+(j)} = I_{m\times m}$$

cancel the operation.

- A: $m\times n$ matrix.

$$\text{rank}(A) \overset{\text{def}}{=\!=} \text{rank}(L_A) = \dim(R(L_A))$$

range of L_A

left multiplication by A

$$L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longmapsto Av$$

Thm: If P is an invertible $m\times m$ matrix

Q is an invertible $n\times n$ matrix

Then $\text{rank}(PA) = \text{rank}(A) = \text{rank}(AQ)$.

Pf: $\text{rank}(PA) = \text{rank}(L_{PA}) = \dim(R(L_{PA}))$

- $L_{PA} = L_P \cdot L_A$. $R(L_{PA}) = R(L_P \cdot L_A) = L_P(R(L_A))$.
- $R(L_A) \subseteq \mathbb{R}^m$, P invertible $\Rightarrow P$ is one-to-one

$\Downarrow$

$\dim R(L_A) = \dim L_P(R(L_A))$.

So $\text{rank}(PA) = \dim(R(L_{PA})) = \dim(L_P(R(L_A))) = \dim R(L_A) = \text{rank}(A)$.

For rank(AQ):

$$\operatorname{rank}(AQ) = \operatorname{rank}(L_{AQ}) = \dim(R(L_{AQ})) = \dim R(L_A \cdot L_Q)$$

$$= \dim A(R(L_Q)) \underset{\uparrow}{=} \dim A(\mathbb{R}^n) = \dim R(A) = \operatorname{rank}(A)$$

Q invertible $\Rightarrow L_Q : \mathbb{R}^n \to \mathbb{R}^n$ invertible $\Rightarrow L_Q$ is onto $\Rightarrow R(L_Q) = \mathbb{R}^n$ ■

Thm: If $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$, then $\operatorname{rank}(A) = \dim \operatorname{Span}\{v_1, v_2, \cdots, v_n\}$.

$\uparrow$ 1st column $\quad$ $\uparrow$ n-th column

Pf: By definition, $\operatorname{rank}(A) = \operatorname{rank}(L_A) = \dim R(L_A)$.

So it suffices to show that $R(L_A) = \operatorname{span}\{v_1, v_2, \cdots, v_n\}$.

Any vector in $R(A)$ is of the form

$$A\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (v_1 v_2 \cdots v_n) \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n.$$

So $R(A)$ consists of linear combinations of column vectors of A.

In other words, $R(A) = \operatorname{span}\{v_1, v_2, \cdots, v_n\}$ ■