- $T: V \to W$ linear transformation

$\beta = \{v_1, v_2, \cdots, v_n\}$ basis for V

$\gamma = \{w_1, w_2, \cdots, w_m\}$ basis for W.

Matrix representation of T w.r.t. β and γ: $[T]_\beta^\gamma = \left([T(v_1)]_\gamma \ [T(v_2)]_\gamma \cdots [T(v_n)]_\gamma \right)$.

- Assume $V = W$, $\beta = \gamma$

Then $[T]_\beta = [T]_\beta^\beta = \left([T(v_1)]_\beta \ [T(v_2)]_\beta \cdots [T(v_n)]_\beta \right)$.

Now we change the basis β to a new basis β'.

We want to find the relation between $[T]_\beta = [T]_\beta^\beta$ and $[T]_{\beta'} = [T]_{\beta'}^{\beta'}$.

To do this, recall the formula for the matrix representation of composition:

$$V \xrightarrow{T} W \xrightarrow{U} Z$$

$$\alpha \qquad \beta \qquad \gamma$$

$$[U \cdot T]_\alpha^\gamma = [U]_\beta^\gamma \cdot [T]_\alpha^\beta$$

identity transformation of V

$Id_V(v) = v, \ \forall v \in V$

Apply this to the composition: $V \xrightarrow{Id_V} V \xrightarrow{T} V \xrightarrow{Id_V} V$

$$\beta' \qquad \beta \qquad \beta \qquad \beta'$$

$$\Rightarrow [T]_{\beta'}^{\beta'} = (Id_V \circ T \circ Id_V)_{\beta'}^{\beta'} = [Id_V]_\beta^{\beta'} \cdot [T]_\beta^\beta \cdot [Id_V]_{\beta'}^\beta$$

Note that $[Id_V]_\beta^{\beta'} \cdot [Id_V]_{\beta'}^{\beta} = [Id_V \cdot Id_V]_{\beta'}^{\beta'} = [Id_V]_{\beta'}^{\beta'} = I_{n\times n}$

So: $[Id_V]_\beta^{\beta'} = \left([Id_V]_{\beta'}^{\beta} \right)^{-1}$.

So We get the formula:

Thm: $[T]_{\beta'}^{\beta'} = \left([Id_V]_{\beta'}^{\beta}\right)^{-1} \cdot [T]_{\beta}^{\beta} \, [Id_V]_{\beta'}^{\beta}$.

Def: $[Id_V]_{\beta'}^{\beta}$ is called the change of coordinate matrix that changes the β'-coordinate to β-coordinate.

Thm: $\forall\, v \in V$, $\quad [v]_{\beta} = [Id_V]_{\beta'}^{\beta} \cdot [v]_{\beta'}$

Proof: Recall the formula for a linear transformation: $\underset{\beta}{V} \xrightarrow{T} \underset{\gamma}{W}$

$$[T(v)]_{\gamma} = [T]_{\beta}^{\gamma} \cdot [v]_{\beta}.$$

Apply this formula to $\underset{\beta'}{V} \xrightarrow{Id_V} \underset{\beta}{V}$ to get:

$$[v]_{\beta} = [Id_V(v)]_{\beta} = [Id_V]_{\beta'}^{\beta} \cdot [v]_{\beta'}. \qquad \blacksquare$$

Ex 1: $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$

$f(x) \mapsto (x-1) f'(x)$.

$$[T]_{\beta} = \Big(\underset{\substack{\shortparallel \\ 0}}{[T(1)]_{\beta}} \quad \underset{\substack{\shortparallel \\ x-1}}{[T(x)]_{\beta}} \quad \underset{\substack{\shortparallel \\ (x-1)\cdot 2x \\ \shortparallel \\ 2x^2-2x}}{[T(x^2)]_{\beta}} \Big) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Choose a new basis $\beta' = \{x^2 - x,\ x^2+1,\ x-1\}$. ($\beta'$ is a basis because it is linearly indep.)

$$[Id]_{\beta'}^{\beta} = \left([x^2-x]_{\beta} \quad [x^2+1]_{\beta} \quad [x-1]_{\beta} \right) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

We calculate its inverse by the process of using elementary transformations:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right)$$

$$\downarrow$$

$$[Id]^{\beta'}_{\beta} = \left([Id]^{\beta}_{\beta'}\right)^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \Leftarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

So

$$[T]_{\beta'} = \left([Id]^{\beta}_{\beta'}\right)^{-1} \cdot [T]_{\beta} \cdot [Id]^{\beta}_{\beta'}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \cdot \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & 4 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

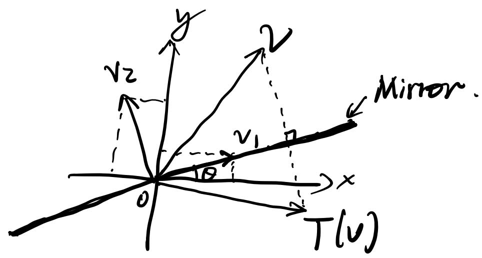
$$= \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

verify: $T(x^2-x) = (x-1)\cdot(2x-1) = 2x^2-3x+1 \overset{\vee}{=} 2\cdot(x^2-x)+0\cdot(x^2+1)-1\cdot(x-1)$

$T(x^2+1) = (x-1)\cdot 2x = 2x^2-2x \overset{\vee}{=} 2\cdot(x^2-x)+0(x^2+1)+0\cdot(x-1)$

$T(x-1) = (x-1)\cdot 1 = x-1 \overset{\vee}{=} 0\cdot(x^2-x)+0\cdot(x^2+1)+1\cdot(x-1)$

Ex3: Reflections revisited.



T = reflection across the mirror $y = (\tan\theta)x$

standard basis $\beta = \{e_1, e_2\}$.

To calculate $[T]_\beta$, we can first find $[T]_{\beta'}$ where

$\beta' = \{v_1, v_2\}$. $|v_1| = |v_2| = 1$, $v_1 \parallel$ mirror, $v_2 \perp$ mirror.

$$[T]_{\beta'} = \left(\underset{\substack{\parallel \\ [v_1]_{\beta'}}}{[T(v_1)]_{\beta'}} \quad \underset{\substack{\parallel \\ [-v_2]_{\beta'}}}{[T(v_2)]_{\beta'}} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$[Id]_{\beta'}^{\beta} = \left(\underset{\substack{\parallel \\ \left[\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right]_\beta}}{[v_1]_\beta} \quad \underset{\substack{\parallel \\ \left[\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\right]_\beta}}{[v_2]_\beta} \right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$[Id]_{\beta}^{\beta'} = \left([Id]_{\beta'}^{\beta} \right)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

So $$[T]_\beta = [T]_\beta^\beta = [Id]_{\beta'}^{\beta} \cdot [T]_{\beta'}^{\beta'} [Id]_{\beta}^{\beta'}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

$[Id_V]_\beta^{\beta'}$ changes β-coordinates to β'-coordinates:

for example, $v = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \Rightarrow [v]_\beta = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$$\Rightarrow [v]_{\beta'} = [Id_V]_\beta^{\beta'} \cdot [v]_\beta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos\theta + 5\sin\theta \\ -3\sin\theta + 5\cos\theta \end{pmatrix}.$$

<u>Check</u>: $(3\cdot\cos\theta + 5\cdot\sin\theta)\cdot v_1 + (-3\sin\theta + 5\cos\theta)\cdot v_2$

$$= (3\cos\theta + 5\sin\theta)\cdot\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + (-3\sin\theta + 5\cos\theta)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 3(\cos^2\theta + \sin^2\theta) \\ 5(\sin^2\theta + \cos^2\theta) \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

<u>Review</u>: • Vector spaces.

• $S \subseteq V$: linearly dependent / independent.

• basis, dimensions.

• $T: V \to W$ linear transformation.

$$\begin{array}{lcc} & V & W \\ & \uparrow & \uparrow \\ \text{bases:} & \beta & \gamma \end{array}$$

$[T]_\beta^\gamma$: matrix representation

$[Tv]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$

$N(T)$, $R(T)$

$\text{nullity}(T) + \text{rank}(T) = \dim V$.

one-to-one, onto, invertible.

composition of linear transformations.