- Let $T: V \to W$ be a linear transformation.

$\beta = \{v_1, \dots, v_n\}$: a basis for V $\quad \#\beta = n = \dim V$

$\gamma = \{w_1, \dots, w_m\}$: a basis for W. $\quad \#\gamma = m = \dim W$.

Matrix representation for T w.r.t. β and γ :

$$[T]_\beta^\gamma = \Big(\underset{\substack{\uparrow \\ \text{coordinates of } T(v_1) \text{ w.r.t. } \gamma}}{[T(v_1)]_\gamma} \; \cdots \; [T(v_n)]_\gamma \Big) \qquad \underset{\substack{\| \\ (\dim W) \times (\dim V)}}{(m \times n)} \text{ matrix}$$

- Composition of linear transformations

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "U \circ T = UT"'] & W \arrow[r, "U"] & Z \end{tikzcd}$$

$(UT)(v) = U(T(v)) \in Z \quad \forall v \in V$. is a linear transformation:

- $(UT)(v_1 + v_2) = U(T(v_1 + v_2)) = U(T(v_1) + T(v_2)) = U(T(v_1)) + U(T(v_2))$
$ = (UT)(v_1) + (UT)(v_2)$

- $(UT)(cv) = U(T(cv)) = U(c \cdot T(v)) = cU(T(v)) = c \cdot (UT)(v)$.

Choose a basis $\alpha = \{v_1, \dots, v_n\}$ for V $\quad \dim V = n$

$\beta = \{w_1, \dots, w_m\}$ for W $\quad \dim W = m$

$\gamma = \{z_1, \dots, z_p\}$ for Z $\quad \dim Z = p$

Thm: $[UT]_\alpha^\gamma = [U]_\beta^\gamma \cdot [T]_\alpha^\beta$ ← matrix multiplication

$p \times n \qquad p \times m \qquad m \times n$

Pf: Denote: $[T]_\alpha^\beta = \underset{\textstyle A}{\underbrace{(a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}}}$, $\quad [U]_\beta^\gamma = \underset{\textstyle B}{\underbrace{[b_{ki}]_{\substack{1\le k\le p \\ 1\le i\le m}}}}$.

$\forall\ v_j \in \alpha$, $j = 1, \dots, n$. we calculate:

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_{i=1}^{m} a_{ij} w_i\right)$$

$$\sum_{i=1}^{m} a_{ij} \sum_{k=1}^{p} b_{ki} z_k = \sum_{i=1}^{m} a_{ij} U(w_i)$$

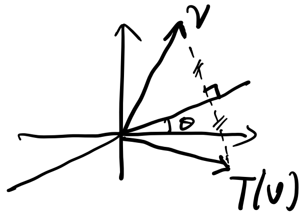
$$\parallel$$

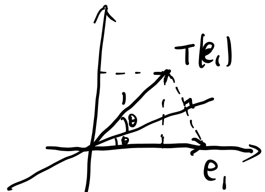
$$\sum_{k=1}^{p} \underbrace{\left(\sum_{i=1}^{m} b_{ki} \cdot a_{ij}\right)} z_k = \sum_{k=1}^{p} (BA)_{kj}\, z_k$$

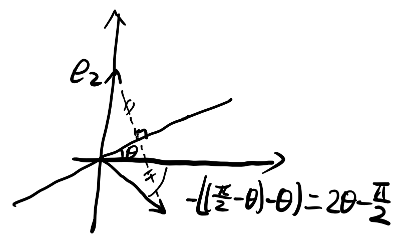
for fixed k and j, this is the inner product of the k-th row of B with the j-th column of A ∎

Example from Geometry. $V=\mathbb{R}^2$, standard basis: $\beta = \{ \underset{e_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{e_2}{\begin{pmatrix}0\\1\end{pmatrix}} \}$

Reflections with respect to the mirror $L = \{ y = (\tan\theta) x \}$





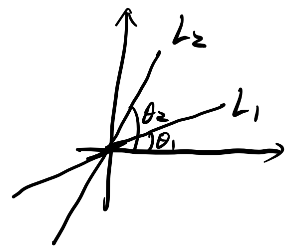


$$T(e_1) = \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} \cos(2\theta - \frac{\pi}{2}) \\ \sin(2\theta - \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}$$

$$\Rightarrow [T]_\beta = [T]_\beta^\beta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

Suppose there are 2 mirrors:



We calculate the matrix representation for the composition of 2 reflections $T_2 \circ T_1$:

$$[T_2 \circ T_1]_\beta = [T_2]_\beta \cdot [T_1]_\beta = \begin{pmatrix} \cos(2\theta_2) & \sin(2\theta_2) \\ \sin(2\theta_2) & -\cos(2\theta_2) \end{pmatrix} \begin{pmatrix} \cos(2\theta_1) & \sin(2\theta_1) \\ \sin(2\theta_1) & -\cos(2\theta_1) \end{pmatrix}$$

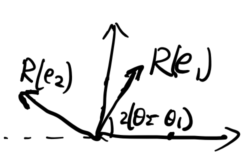
$$\parallel$$

$$\begin{pmatrix} \cos(2(\theta_2-\theta_1)) & -\sin(2(\theta_2-\theta_1)) \\ \sin(2(\theta_2-\theta_1)) & \cos(2(\theta_2-\theta_1)) \end{pmatrix} = \begin{pmatrix} \cos(2\theta_2)\cos(2\theta_1) + \sin(2\theta_2)\sin(2\theta_1) & \cos(2\theta_2)\sin(2\theta_1) - \sin(2\theta_2)\cos(2\theta_1) \\ \sin(2\theta_2)\cos(2\theta_1) - \cos(2\theta_2)\sin(2\theta_1) & \sin(2\theta_2)\sin(2\theta_1) + \cos(2\theta_2)\cos(2\theta_1) \end{pmatrix}$$

$\uparrow$

matrix representation of rotation by the angle $2(\theta_2-\theta_1)$

$R = R_{2(\theta_2-\theta_1)}$



$\Rightarrow$ Fact: Composition of 2 reflections is a rotation.

Example from Calculus: $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$

Integration: $f \mapsto \int_0^x f(t)\,dt$

$\beta = \{1, x, x^2\}$ basis for $P_2(\mathbb{R})$. $\gamma = \{1, x, x^2, x^3\}$ basis for $P_3(\mathbb{R})$.

$$[T]_\beta^\gamma = \Big(\underset{\substack{\parallel \\ [x]_\gamma}}{[T(1)]_\gamma} \quad \underset{\substack{\parallel \\ [\frac{1}{2}x^2]_\gamma}}{[T(x)]_\gamma} \quad \underset{\substack{\parallel \\ [\frac{1}{3}x^3]_\gamma}}{[T(x^2)]_\gamma} \Big) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = A$$

Differentiation $U: P_3(\mathbb{R}) \to P_2(\mathbb{R})$. $f \mapsto f'$

$$[U]_\gamma^\beta = \Big(\underset{\substack{\parallel \\ 0}}{[U(1)]_\beta} \quad \underset{\substack{\parallel \\ [1]_\beta}}{[U(x)]_\beta} \quad \underset{\substack{\parallel \\ [2x]_\beta}}{[U(x^2)]_\beta} \quad \underset{\substack{\parallel \\ [3x^2]_\beta}}{[U(x^3)]_\beta} \Big) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = B$$

Composition: • $[UT]_\beta^\beta = [U]_\gamma^\beta \cdot [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$UT(f) = U\left(\int_0^x f(t)\,dt\right) = \frac{d}{dx}\int_0^x f(t)\,dt = f, \quad \forall f \in P_2(\mathbb{R})$$

So UT is the identity transformation.

• $[TU]_\gamma^\gamma = [T]_\beta^\gamma \cdot [U]_\gamma^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$TU(f) = T(f') = \int_0^x f'(t)\,dt = f(x) - f(0)$$

If $f = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3$, then $TU(f) = a_1 \cdot x + a_2 \cdot x^2 + a_3 x^3$.

So TU is a projection.

$T: V \to W$ a linear transformation.

<u>Def</u>: • T is one-to-one if T is injective:

$$v_1 \neq v_2 \Rightarrow T(v_1) \neq T(v_2)$$

equivalently $T(v_1) = T(v_2) \Rightarrow v_1 = v_2$.

• T is onto if T is surjective, i.e. $\underbrace{R(T)}_{\substack{\text{range of } T \\ = \{T(v):\, v \in V\}}} = W$:

$$\forall w \in W, \exists v \in V \text{ s.t. } T(v) = w.$$

<u>Thm</u>: T is one-to-one if and only if $\underbrace{N(T)}_{= \{v \in V:\, T(v) = 0\}} = \{0\}$.

<u>Proof</u>: "if" Assume $N(T) = \{0\}$.

Then $T(v_1) = T(v_2) \Rightarrow \underbrace{T(v_1) - T(v_2)}_{= T(v_1 - v_2)} = 0 \Rightarrow v_1 - v_2 \in N(T)$

$\Downarrow$

$v_1 - v_2 = 0$ i.e. $v_1 = v_2$ ✓

"only if" Assume T is one-to-one. Then

$v \in N(T) \Rightarrow T(v) = 0 = T(0) \Rightarrow v = 0$. So $N(T) = \{0\}$ ✓

∎