A linear transformation is determined by its values on a basis:

<u>Thm</u>: Let $\beta = \{v_1, \cdots, v_n\}$ be a basis for a vector space V

Let $w_1, \cdots, w_n$ be n vectors of W (which may not be distinct).

Then there exists exactly one linear transformation $T: V \to W$ that satisfies

$T(v_i) = w_i$. $i = 1, \cdots, n$.

<u>Pf</u>: Any $v \in V$ can be written as a linear combination of β in a unique way:

$v = a_1 v_1 + \cdots + a_n v_n$. with $a_1, \cdots, a_n \in F$

Then $T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 \cdot T(v_1) + \cdots + a_n \cdot T(v_n)$

$T(v) = a_1 w_1 + \cdots + a_n \cdot w_n$. $(*)$

So $T(v)$ is completely determined by $T(v_i) = w_i$.

Conversely, $(*)$ defines a function $T: V \to W$ that is linear (by a direct verification)

<u>Ex</u>: If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear and satisfies $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$, $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$

Then we can calculate $T(v)$ for any $v \in \mathbb{R}^2$ because

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

For example, to calculate $T\left(\begin{pmatrix}2\\3\end{pmatrix}\right)$: First write $\begin{pmatrix}2\\3\end{pmatrix}$ as a linear comb:

$$\begin{pmatrix}2\\3\end{pmatrix} = a_1 \begin{pmatrix}1\\0\end{pmatrix} + a_2 \begin{pmatrix}1\\1\end{pmatrix} \Leftrightarrow \begin{cases} a_1 + a_2 = 2 \\ a_2 = 3 \end{cases} \Rightarrow \begin{cases} a_1 = -1 \\ a_2 = 3. \end{cases}$$

So $T\begin{pmatrix}2\\3\end{pmatrix} = a_1 T\begin{pmatrix}1\\0\end{pmatrix} + a_2 T\begin{pmatrix}1\\1\end{pmatrix} = -1 \cdot \begin{pmatrix}1\\4\end{pmatrix} + 3 \cdot \begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}5\\11\end{pmatrix}$.

- Fix a basis $\beta = \{v_1, \cdots, v_n\}$ for V. $v \in V$:

$$v = a_1 v_1 + \cdots + a_n v_n \rightsquigarrow [v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \begin{pmatrix} \text{coordinate vector of} \\ v \text{ relative to } \beta \end{pmatrix}$$

- Matrix representation of a linear transformation $T: V \to W$

with respect to a basis $\beta = \{v_1, \cdots, v_n\}$ for V
a basis $\gamma = \{w_1, \cdots, w_m\}$ for W.

T is determined by $T(v_j)$, $j = 1, \cdots, n$:

$$T(v_j) = a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_m = (w_1\ w_2 \cdots w_m) \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

$$\rightsquigarrow [T]_\beta^\gamma = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad m \times n \text{ matrix.}$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$[T(v_1)]_\gamma \ [T(v_2)]_\gamma \cdots [T(v_n)]_\gamma$$

$$T(\underbrace{v_1\ v_2 \cdots v_n}_{\beta}) = (\underbrace{w_1\ w_2 \cdots w_m}_{\gamma}) [T]_\beta^\gamma.$$

$$m = \dim W = \#\gamma, \quad n = \dim V = \#\beta.$$

<u>Ex cont.</u> $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}$, $T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\5\end{pmatrix}$.

$\beta=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ standard basis

$\gamma=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$

- Calculate $[T]_\beta^\beta$: $\left[\begin{pmatrix}a\\b\end{pmatrix}\right]_\beta=\begin{pmatrix}a\\b\end{pmatrix}$

$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix}=-T\begin{pmatrix}1\\0\end{pmatrix}+T\begin{pmatrix}1\\1\end{pmatrix}=-\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = T\left(-\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}\right)$, $\quad -\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

$\rightsquigarrow [T]_\beta^\beta=\begin{pmatrix}1&1\\4&1\end{pmatrix}$.

$\begin{pmatrix}a_1+a_2\\a_2\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$, $\begin{cases}a_1+a_2=1\\a_2=4\end{cases} \Downarrow$

- Calculate $[T]_\beta^\gamma$: $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}=a_1\begin{pmatrix}1\\0\end{pmatrix}+a_2\cdot\begin{pmatrix}1\\1\end{pmatrix}$ $\quad \begin{cases}a_1=-3\\a_2=4\end{cases}$

$T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}=0\cdot\begin{pmatrix}1\\0\end{pmatrix}+1\cdot\begin{pmatrix}1\\1\end{pmatrix}$

So $[T]_\beta^\gamma=\begin{pmatrix}-3&0\\4&1\end{pmatrix}$

- Calculate $[T]_\gamma^\beta$: $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}$, $T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\5\end{pmatrix}$

$\Rightarrow [T]_\gamma^\beta=\begin{pmatrix}1&2\\4&5\end{pmatrix}$

Ex:

$$T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$$
$$\rotatebox{90}{$\in$} \qquad\qquad$$
$$f \longmapsto f'$$

$\beta = \{1, x, x^2\}$

$T(1) = 1' = 0 \quad \rightsquigarrow [T(1)]_\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$T(x) = x' = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 \rightsquigarrow [T(x)]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$T(x^2) = (x^2)' = 2x = 0\cdot 1 + 2\cdot x + 0 x^2 \rightsquigarrow [T(x^2)]_\beta = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$$\Longrightarrow [T]_\beta^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Ex:

$$T: P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$$
$$f \longmapsto \int_0^x f(t)\,dt$$

$\underbrace{\beta}_{\text{basis of } P_2(\mathbb{R})} = \{1, x, x^2\}$, $\quad \underbrace{\gamma}_{\text{basis of } P_3(\mathbb{R})} = \{1, x, x^2, x^3\}$

$$T(1) = \int_0^x 1\, dt = x = 0\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3$$

$$\rightsquigarrow \quad [T(1)]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(x) = \int_0^x t\, dt = \frac{x^2}{2} = 0\cdot 1 + 0\cdot x + \frac{1}{2}\cdot x^2 + 0\cdot x^3$$

$$\rightsquigarrow \quad [T(x)]_\gamma = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$T(x^2) = \int_0^x t^2\, dt = \frac{x^3}{3} = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + \frac{1}{3}\cdot x^3$$

$$\rightsquigarrow \quad [T(x^2)]_\gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}$$

$$\Rightarrow \quad [T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Composition:

$$\begin{tikzcd}
 & U^k \arrow[r, "S"] \arrow[rr, bend right, "T\circ S"'] & V^n \arrow[r, "T"] & W^m \\
\text{basis:} & \alpha \arrow[u] & \beta \arrow[u] & \gamma \arrow[u] \\
\#\text{basis} & k & n & m
\end{tikzcd}$$

$$\underset{m\times k}{[T\circ S]_\alpha^\gamma} = \underset{m\times n}{[T]_\beta^\gamma}\ \underset{n\times k}{[S]_\alpha^\beta}$$