Let V, W be vector spaces$/F$

Def: A function $T: V \to W$ is called a linear transformation if it satisfies the following 2 conditions:

1. $T(x+y) = T(x) + T(y)$, $\forall x, y \in V$. (T preserves the addition)
2. $T(c \cdot x) = c \cdot T(x)$, $\forall x \in V, c \in F$ (T preserves the scalar multiplication)

Immediate Properties: ①: $T(\overset{0_V}{\overset{\shortparallel}{0}}) = \overset{0_W}{\overset{\shortparallel}{0}}$: Pick any $x \in V$.

$$T(x) + T(0) = T(x+0) = T(x) = T(x) + 0 \xRightarrow{\text{Cancellation rule}} T(0) = 0.$$

② $T(x-y) = T(x) - T(y)$

$$\underset{\substack{\shortparallel \\ T(x+(-y))}}{T(x-y)} = \underset{\shortparallel}{T(x) - T(y)}$$

$$T(x+(-y)) = T(x) + T((-1)\cdot y) = T(x) + (-1)\cdot T(y)$$

③ $T(a_1 x_1 + \cdots + a_n x_n) = a_1 \cdot T(x_1) + \cdots + a_n \cdot T(x_n)$

Ex: Consider functions from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$T_1(a_1, a_2) = (a_1 + a_2, 3a_2 - a_1)$ is linear

$T_2(a_1, a_2) = (a_1 + 1, a_2)$ is not linear ($T_2(0,0) \neq (0,0)$)

$T_3(a_1, a_2) = (a_1^2, 0)$ not linear: it does not preserve addition

$$\left(\begin{array}{l} T_3((a_1, a_2) + (b_1, b_2)) = T_3(a_1 + b_1, a_2 + b_2) = ((a_1+b_1)^2, 0) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \# \\ T_3(a_1, a_2) + T_3(b_1, b_2) = (a_1^2, 0) + (b_1^2, 0) = (a_1^2 + b_1^2, 0) \end{array} \right)$$

Ex: $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ is linear:

$$\begin{array}{ccc} f & \longmapsto & f' \\ \| & & \| \\ a_0 + a_1 x + a_2 x^2 & & a_1 + 2a_2 x \end{array}$$

$T(f+g) = f' + g' = T(f) + T(g)$

$T(cf) = (cf)' = cf' = c \cdot T(f)$

Ex: $S: P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ is also linear.

$$\begin{array}{ccc} f & \longmapsto & \int_0^x f(t)\,dt \\ \| & & \| \\ a_0 + a_1 x + a_2 x^2 & & a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 \end{array}$$

- $S(f+g) = \int_0^x (f+g)(t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt = S(f) + S(g)$
- $S(cf) = \int_0^x (cf)(t)\,dt = c\int_0^x f(t)\,dt = c \cdot S(f)$.

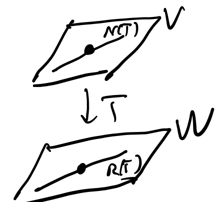
Def: Null space of a linear transformation $T: V \to W$:

$$N(T) = \{v \in V : Tv = 0\} \subseteq V$$

Range of T: $R(T) = \{T(v) \in W : v \in V\} \subseteq W$.

Thm: $N(T)$ is a (linear) subspace of V

$R(T)$ is a (linear) subspace of W.



Pf: Verify that $N(T)$ and $R(T)$ are closed w.r.t. addition and scalar mult.

Def: Nullity of T: $\text{nullity}(T) = \dim N(T)$

Rank of T: $\text{rank}(T) = \dim R(T)$

We always assume $\dim V < +\infty$, $\dim W < +\infty$ so that $\text{nullity}(T) < +\infty$, $\text{rank}(T) < +\infty$.

Ex: For $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$.
$f \mapsto f'$

- $N(T) = \{f \in P_2(\mathbb{R}) : f' = 0\} = \{a_0 : a_0 \in \mathbb{R}\} = \mathbb{R}$

nullity$(T) = \dim N(T) = 1$.

- $R(T) = \{a_0 + 2a_1 \cdot x : a_0, a_1 \in \mathbb{R}\} = \text{span}\{1, x\} = P_1(\mathbb{R})$

rank$(T) = \dim R(T) = 2$.

nullity(T) + rank$(T) = 1+2 = 3 = \dim P_2(\mathbb{R})$.

Ex: For $S: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ $f \mapsto \int_0^x f(t)\,dt$

$N(T) = \{a_0 + a_1 \cdot x + a_2 \cdot x^2 : a_0 x + a_1 \cdot \frac{x^2}{2} + a_2 \cdot \frac{x^3}{3} = 0\}$

$= \{a_0 + a_1 \cdot x + a_2 \cdot x^2 : a_0 = a_1 = a_2 = 0\} = \{0\}$

nullity$(T) = \dim\{0\} = 0$.

$R(T) = \{a_0 \cdot x + a_1 \cdot \frac{x^2}{2} + a_2 \cdot \frac{x^3}{3} : a_0, a_1, a_2 \in \mathbb{R}\}$

$= \text{span}\{x, x^2, x^3\}$

rank$(T) = \dim R(T) = 3$.

nullity(T) + rank$(T) = 0 + 3 = 3$.

Thm: Let $T: V \to W$ be any linear transformation between vector spaces of finite dimensions.

Then we always have the identity:

$$\text{nullity}(T) + \text{rank}(T) = \dim V.$$

Pf: Choose a basis $\beta = \{v_1, \cdots, v_m\}$ of $N(T)$.

Because β is a linearly indep. subset of V, by applying Replacement Thm

β can be extended to a basis $\beta' = \{v_1, \cdots, v_m, u_1, \cdots, u_k\}$ of V.

We need to show that $\dim(V) - \text{nullity}(T) = (m+k) - m = k$ is equal to $\text{rank}(T)$.

Consider the subset $\gamma = \{T(u_1), \cdots, T(u_k)\}$ of W.

Just need to show that γ is a basis of $R(T)$ to get $k = \text{rank}(T) = \dim R(T)$.

- show γ is linearly independent:

Assume $a_1 T(u_1) + \cdots + a_k T(u_k) = 0$. Then $a_1 u_1 + \cdots + a_k u_k \in N(T)$

$\overset{T \text{ is linear}}{=} T(a_1 u_1 + \cdots + a_k u_k)$

Because $\beta = \{v_1, \cdots, v_m\}$ spans $N(T)$, $a_1 u_1 + \cdots + a_k u_k = b_1 v_1 + \cdots + b_m v_m$.

But $\{u_1, \cdots, u_k, v_1, \cdots, v_m\} = \beta'$ is linearly independent (it is a basis of V)

So $a_1 = \cdots = a_k = 0 = b_1 = \cdots = b_m$.

So there is only trivial relation for γ.

- Show γ spans $R(T)$. Pick any $T(v) \in R(T)$ with $v \in V$.

Because β' is a basis for V, $v = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_k u_k$ for some $a_1, \cdots, a_m, b_1, \cdots, b_k \in F$

Then $T(v) = a_1 T(v_1) + \cdots + a_m T(v_m) + b_1 T(u_1) + \cdots + b_k T(u_k)$

$= b_1 T(u_1) + \cdots + b_k T(u_k)$ ($v_i \in N(T) \Rightarrow T(v_i) = 0$, $i = 1, \cdots, m$)

So $T(v) \in \text{Span}(\gamma)$, $\forall v \in V \Rightarrow R(T) \subseteq \text{Span}(\gamma)$ ∎