Thm: If $V = \mathrm{Span}(S)$ and $\#S$ is finite. Then

$\uparrow$ number of vectors in S

There is a subset of S that is a basis for V.

Pf: Assume $S = \{v_1, \dots, v_n\}$ $v_i \neq v_j$ if $i \neq j$.

We construct a subset of S by the following process:

step 1: if $v_1 \neq 0$, then set $S_1 = \{v_1\}$, Otherwise set $S_1 = \phi$.

step 2: if $v_2 \notin \mathrm{Span}(S_1)$, then set $S_2 = S_1 \cup \{v_2\}$. Otherwise set $S_2 = S_1$.

step 3: if $v_3 \notin \mathrm{Span}(S_2)$, then set $S_3 = S_2 \cup \{v_3\}$, Otherwise set $S_3 = S_2$.

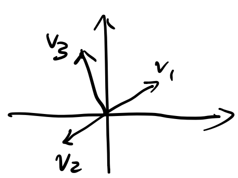
...

step n: if $v_n \notin \mathrm{Span}(S_{n-1})$, then set $S_n = S_{n-1} \cup \{v_n\}$. Otherwise set $S_n = S_{n-1}$

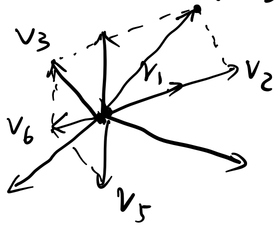
At each step, we get a linearly independent subset. and $\mathrm{span}(S_n) = \underbrace{\mathrm{Span}(S)}_{= V}$

So S_n is a basis for V.

Ex:



$S = \{v_1, v_2, v_3\} \rightsquigarrow S_1 = \{v_1\}, S_2 = \{v_1\}, \underbrace{S_3 = \{v_1, v_3\}}_{\text{basis for } \mathbb{R}^2}$



$S = \{v_1, v_2, v_3, v_4, v_5\}$

$S_1 = \{v_1\}$, $S_2 = \{v_1\}$, $S_3 = \{v_1, v_3\}$. $S_4 = \{v_1, v_3\}$

$S_5 = \{v_1, v_3, v_5\}$, $\underbrace{S_6 = \{v_1, v_3, v_5\}}_{\text{a basis for } \mathbb{R}^3}$

We assume that V is spanned by a finite subset.

<u>Thm A</u>: Any two basis for a vector space contain the same number of vectors.

We can now define $\dim V = \# \beta$ if β is a basis for V.
$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\# \beta'}$ for any other basis β'

V is spanned by a finite subset $\Rightarrow \dim V < +\infty$ (finite dimension)

<u>Thm B</u>. 1. If $V = \mathrm{Span}(S)$, then $\# S \geqslant \dim V$

2. If $V = \mathrm{Span}(S)$ and $\# S = \dim V$, then S is a basis

3. If S is linearly independent and $\# S = \dim V$, then S is a basis.

4. If S is linearly independent then $\# S \leq \dim V$.
Moreover S can be extended to become a basis for V.
(i.e. $\exists u_1, \cdots, u_k \in V$ s.t. $S \cup \{u_1, \cdots, u_k\}$ is a basis)

<u>Ex</u>: We can use Thm B.3 to verify whether a subset is a basis or not

For example. let $S = \left\{ \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 8 \\ 2 \end{pmatrix} \right\}$. $\# S = 3$.

To check whether S is a basis or not, just need to check whether S is linearly independent

S is linearly independent iff the following system has only 0 solution.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} + a_3 \begin{pmatrix} -3 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -3 \\ 3 & -4 & 8 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Use Gauss-Elimination method:

$$\begin{pmatrix} -1 & 2 & -3 \\ 3 & -4 & 8 \\ 1 & -3 & 2 \end{pmatrix} \xrightarrow[\text{①}+\text{③}]{\text{①}\cdot 3+\text{②}} \begin{pmatrix} -1 & 2 & -3 \\ 0 & 2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[\text{②}\leftrightarrow\text{③}]{\text{③}\cdot 2+\text{②}} \begin{pmatrix} -1 & 2 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{pmatrix} \Rightarrow \text{no free variables}$$

$\Downarrow$

only zero solution

$\Updownarrow$

$$\begin{cases} -a_1+a_2-3a_3=0 \\ -a_2-a_3=0 \\ -3a_3=0 \end{cases} \Rightarrow a_3=a_2=a_1=0.$$

So S is linearly indep. and

$$\# S = 3 = \underbrace{\dim \mathbb{R}^3}_{\substack{\shortparallel \\ \#\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}}}$$

By Thm B.3, S is indeed a basis.

<u>Ex</u>: $V = P_2(\mathbb{R})$,

$$S = \{-1+3x+x^2,\ 2-4x-3x^2,\ -1+5x\}$$

Check whether S is lin. indep or not: Find if there are nontrivial rep. of 0:

$$0 = a_1\cdot(-1+3x+x^2) + a_2(2-4x-3x^2) + a_3(-1+5x)$$

$$= (-a_1+2a_2-a_3) + (3a_1-4a_2+5a_3) + (a_1-3a_2)x^2$$

$$\Leftrightarrow \begin{pmatrix} -1 & 2 & -1 \\ 3 & -4 & 5 \\ 1 & -3 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & -1 \\ 3 & -4 & 5 \\ 1 & -3 & 0 \end{pmatrix} \xrightarrow[\text{①}+\text{③}]{\text{①}\cdot 3+\text{②}} \begin{pmatrix} -1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\exists$ free variable $\Rightarrow$ $\exists$ non zero solution

$\Rightarrow S$ is not linearly indep. $\Rightarrow S$ is not a basis for V.

Mother Thm (Replacement Theorem).

Assume: • $V = \mathrm{Span}(G)$, $G = n$

• L is linearly independent subset of V. $\# L = m$.

Then conclusions: • $n \geq m$

• $\exists$ $(n-m)$ vectors (distinct) $v_1, \cdots, v_{n-m}$ of G

s.t. $L \cup \{v_1, \cdots, v_{n-m}\}$ spans V.

In otherwords, if we write $G = \{v_1, \cdots, v_{n-m}, u_1, \cdots, u_m\}$.

Then if we replace $\{u_1, \cdots, u_m\}$ of G by L, we still get a spanning subset $\{v_1, \cdots, v_{n-m}\} \cup L$.

The Replacement Thm implies Thm A, Thm B.1–4.

For example: Pf. of Thm A: Let β_1, β_2 be two bases.

$G \leftarrow \beta_1$
$L \leftarrow \beta_2$ $\Longrightarrow$ $\# \beta_1 \geq \# \beta_2$. Changing the row of β_1, β_2. Get $\# \beta_2 \geq \# \beta_1$

so $\# \beta_1 = \# \beta_2$.

pf of Thm B.4: S linearly independent. Let β be a basis.

$G \leftarrow \beta$
$L \leftarrow S$ $\xrightarrow[\text{Thm}]{\text{Replacement}}$ $\underset{n}{\#\beta} \geq \underset{m}{\# S}$. and $\exists \{v_1, \cdots, v_{n-m}\} \subseteq \beta$ s.t.

$S \cup \{v_1, \cdots, v_{n-m}\}$ is a basis.

Proof of the Replacement Thm: Induction on m.

- when $m=0$, $\underset{\substack{\uparrow \\ \text{empty set.}}}{L=\phi}$, $\phi \cup \underset{\substack{\uparrow \\ \#G=n=n-0}}{G}=G$ spans V $\checkmark$.

- Assume that the Thm is already proved for $m-1$, for $m \geq 1$. Want to prove the conclusion when $\#L=m$.

  Write $L=L_1 \cup \{u\}$ with $\#L_1=m-1$, and $u \notin \text{Span}(L_1)$.

  By the induction assumption, there exist $\{v_1, \cdots, v_{n-(m-1)}\} \subseteq G$ s.t. $L_1 \cup \{v_1, \cdots, v_{n-m+1}\}$ spans V. $n \geq m-1$.

  In particular, $u \in \text{Span}(L_1 \cup \{v_1, \cdots, v_{n-m+1}\})$. So $\exists$ $u_1, \cdots, u_k \in L_1$ s.t. $a_1, \cdots, a_k \in F$ and $b_1, \cdots, b_{n-m+1} \in F$

  $$u = a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_{n-m+1}v_{n-m+1}$$

  if $n-(m-1)=0$, then $u \in \text{Span}\{u_1, \cdots, u_k\} \subseteq \text{Span}(L_1)$ contradicting $u \notin \text{Span}(L_1)$

  so $n-(m-1)>0 \Rightarrow n \geq m$. and $b_1, \cdots, b_{n-m+1}$ not all 0.

  By reordering $v_1, \cdots, v_{n-m+1}$, we can assume that $b_{n-m+1} \neq 0$.

  then $v_{n-m+1} = b_{n-m+1}^{-1}(-u + a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_{n-m}v_{n-m})$

  implies $\underline{v_{n-m+1}} \in \text{Span}\{u, u_1, \cdots, u_k, v_1, \cdots, v_{n-m}\}$

  so $\underset{\substack{\| \\ V}}{\text{Span}\{L_1 \cup \{v_1, \cdots, \underline{v_{n-m+1}}\}\}} \subseteq \underset{\substack{\| \\ \text{Span}\{L \cup \{v_1, \cdots, v_{n-m}\}\}}}{\text{Span}\{L_1 \cup \{u\} \cup \{v_1, \cdots, v_{n-m}\}\}}$

  $\Rightarrow \text{Span}(L \cup \{v_1, \cdots, v_{n-m}\}) = V$ as wanted ∎