1.5 <u>Def</u>: A subset S of a vector space V is called linearly dependent if there exists a finite number of distinct vectors $u_1, \cdots, u_n \in S$ and scalars $a_1, \cdots, a_n$, not all zero s.t.

$$a_1 u_1 + \cdots + a_n u_n = 0.$$

In other words, there is a non-trivial representation of 0 as a linear combination of vectors in S.

- Trivial representation of the 0 vector: $0 = 0 \cdot u_1 + \cdots + 0 \cdot u_n$

<u>Def</u>: A subset S is linearly independent if it is not linearly dependent.
In other words, there is no nontrivial representation of 0 as a linear combination of vectors in S.

<u>Ex</u>: If S contains the 0 vector, then it is always linearly dependent because $\exists$ nontrivial rep.: $0 = a \cdot 0 \quad a \neq 0 \in F$.

<u>Ex</u>: $S = \{v\}$ is linearly dependent iff $v = 0$:
Because if there is a nontrivial representation of 0:

$$0 = a \cdot v \quad a \neq 0, \text{ then } \underset{\underset{0}{\shortparallel}}{a^{-1} \cdot 0} = a^{-1} a \cdot v = 1 \cdot v = v.$$

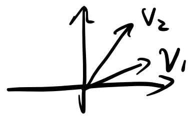
<u>Ex</u>: $S = \{\underset{\underset{0}{\not\shortparallel}}{v_1}, \underset{\underset{0}{\not\shortparallel}}{v_2}\}$ is linearly dependent iff v_2 is a $\overset{\text{scalar}}{\text{multiple}}$ of v_1

$$a_1 \cdot v_1 + a_2 \cdot v_2 = 0, \ \underset{v_1 \neq 0,\ v_2 \neq 0}{(a_1, a_2) \neq (0,0)} \Rightarrow a_1 \neq 0, a_2 \neq 0$$
$$\Rightarrow v_2 = -a_2^{-1} a_1 \cdot v_1$$

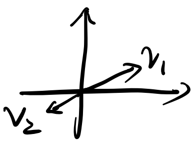
Ex: $S=\{v_1, v_2, v_3\}$ $(v_1 \neq 0,\ v_2 \neq 0,\ v_3 \neq 0)$ is linearly dependent iff

either $\{v_1,v_2\}$ is linearly dependent or $v_3 \in \mathrm{Span}\{v_1, v_2\}$.

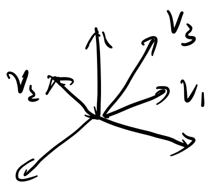
Ex:



$\{v_1,v_2\}$ linearly indep.



$\{v_1,v_2\}$ linearly dependent



$\{v_1,v_2,v_3\}$ linearly dependent iff $v_3 \in \mathrm{Span}\{v_1,v_2\}$

(Assume $v_2 \nparallel v_1$) $(v_2 \neq 0,\ v_1 \neq 0)$

Ex: Determine whether the subset $S=\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\-5\\2\end{pmatrix}\right\} \subset \mathbb{R}^3$ is linearly dependent or not.

Sol: S is linearly dependent iff there is a non-trivial rep. of 0:

$$a_1\begin{pmatrix}1\\-1\\2\end{pmatrix} + a_2\begin{pmatrix}1\\1\\2\end{pmatrix} + a_3\begin{pmatrix}1\\-5\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

iff $\begin{pmatrix}1 & 1 & 1\\-1 & 1 & -5\\2 & 2 & 2\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ has a nonzero solution.

We can use Gauss elimination to determine whether there are nonzero solutions or not:

$$\begin{pmatrix}1 & 1 & 1\\-1 & 1 & -5\\2 & 2 & 2\end{pmatrix} \xrightarrow[\text{①}\cdot(-2)+\text{③}]{\text{①}+\text{②}} \begin{pmatrix}1 & 1 & 1\\0 & 2 & -4\\0 & 0 & 0\end{pmatrix} \xrightarrow{\text{②}/2} \begin{pmatrix}1 & 1 & 1\\0 & 1 & -2\\0 & 0 & 0\end{pmatrix} \xrightarrow{\text{②}(-1)+\text{①}} \begin{pmatrix}1 & 0 & 3\\0 & 1 & -2\\0 & 0 & 0\end{pmatrix}$$ free var.

$\exists$ free variables $\Rightarrow \exists$ nonzero solutions

$$\begin{cases} a_1 = -3a_3 \\ a_2 = 2a_3 \end{cases}$$

So there are nonzero solutions $\Rightarrow \exists$ nontrivial rep. of $0 \left(= \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$

$\Leftarrow$ S is linearly dependent.

Thm: Let S be a linearly independent subset of V.

$v \in V \setminus S$ (i.e. $v \in V$ but $v \notin S$). Then

$S \cup \{v\}$ is linearly dependent if and only if (iff) $v \in \mathrm{Span}(S)$.

Pf: "if": if $v \in \mathrm{Span}(S)$, then $v = a_1 u_1 + \cdots + a_n u_n$. $u_1, \dots, u_n \in S$.

Then $0 = \underbrace{-v}_{\substack{\shortparallel \\ (-1)\cdot v}} + a_1 u_1 + \cdots + a_n u_n$ is a nontrivial rep. of 0 as

a linear combination of vectors in $S \cup \{v\}$. So $S \cup \{v\}$ is lin. dep.

"only if": Assume $S \cup \{v\}$ is linearly dependent, then there is

a nontrivial representation of 0:

$0 = a_1 u_1 + \cdots + a_n u_n$, with distinct $u_1, \dots, u_n \in S \cup \{v\}$, $a_1, \dots, a_n$ not zero.

Case 1: $u_i \neq v$, for $i = 1, \dots, n$. Then there is a nontrivial rep. of 0 as a linear combination of vectors in S. This contradicts the assumption that S is linearly independent. Case 1 can't happen.

Case 2: By re-ordering u_i's, we can assume $u_1 = v$. So we have.

$0 = a_1 \cdot v + a_2 u_2 + \cdots + a_n u_n$ with $a_1 \neq 0$

Then $v = -a_1^{-1}(a_2 u_2 + \cdots + a_n u_n)$

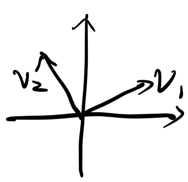
$= -(a_1^{-1} a_2) u_2 - \cdots - (a_1^{-1} a_n) \cdot u_n \in \mathrm{Span}(S)$ ∎

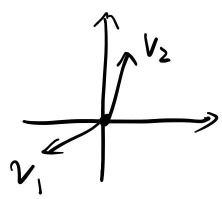
1.6

Def: A basis β for a vector space V is a subset that satisfies:

1. β is linearly independent.
2. β generates (spans) V, i.e. $\mathrm{Span}(\beta) = V$.

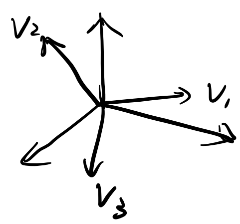
Ex: basis for $\mathbb{R}^2$: $\beta = \{v_1, v_2\}$ $\quad v_1 \nparallel v_2$:





- Standard basis for $\mathbb{R}^3$: $\left\{ \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{=e_1}, \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{=e_2}, \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{=e_3} \right\}$

General basis: $\{v_1, v_2, v_3\}$ linearly independent.



Ex: $P_2(\mathbb{R}) = \{\text{polynomials of deg} \leq 2\}$

A basis: $\beta = \{1, x, x^2\}$:

- $\mathrm{Span}(\beta) = P_2(\mathbb{R})$: $a_0 + a_1 x + a_2 x^2 = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$
- linearly independent: $a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = 0 = 0 + 0 \cdot x + 0 \cdot x^2$

$\Rightarrow a_0 = a_1 = a_2 = 0$. (no nontrivial rep. of 0)

Thm: A subset β is a basis for V iff any $v \in V$ can be uniquely expressed as a linear combination of vectors of β.

Pf: "if" Assume: any $v \in V$ can be uniquely expressed as a linear combination of vectors of β. Show that β satisfies:

- β spans V (True by the Assumption)
- β is linearly independent:

$$\left.\begin{aligned} 0 &= a_1 \cdot u_1 + \cdots + a_n \cdot u_n \\ 0 &= 0 \cdot u_1 + \cdots + 0 \cdot u_n \quad \text{(trivial rep.)} \end{aligned}\right\} \xRightarrow[\text{expressed}]{\text{"uniquely"}} a_1 = 0, \dots, a_n = 0$$

So no nontrivial rep. of 0

"only if" Assume β is a basis. Then

- $\mathrm{Span}(\beta) = V \Rightarrow$ any $v \in V$ is a linear combination of vectors of β.
- Assume there are two representation of $v \in V$:

$$v = a_1 \cdot u_1 + \cdots + a_n \cdot u_n = b_1 u_1 + \cdots + b_n u_n \qquad \begin{matrix} u_1, \dots, u_n \in \beta \\ \text{distinct} \end{matrix}$$

Then $(b_1 - a_1) u_1 + \cdots + (b_n - a_n) u_n = 0$.

Because β is linearly independent, $b_1 = a_1, \dots, b_n = a_n$.

So the representation of v as a linear combination of vectors of β is unique. ∎