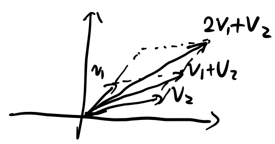
V: a vector space over a field F

<u>Def</u>: $v \in V$ is a linear combination of vectors of S if
$\exists$ a finite numbers of vectors $u_1, \cdots, u_n \in S$ and
scalars $a_1, \cdots, a_n \in F$
s.t. $v = a_1 u_1 + \cdots + a_n u_n$.

<u>Ex</u>: $V = \mathbb{R}^2$



<u>Ex</u>: $V = \mathbb{R}^3$

$$3\begin{pmatrix}1\\-1\\2\end{pmatrix} - 2\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}1\\-5\\2\end{pmatrix}$$

<u>Ex</u>: $V = P(\mathbb{R}) = \{\text{polynomials with real coefficients}\}$

$$1-5x+2x^2 = 3\cdot(1-x+2x^2) - 2\cdot(1+x+2x^2)$$

<u>Def</u>: Let S be a non-empty subset of V.
Then $\mathrm{span}(S) = \{\text{linear combinations of vectors in } S\}$

$$= \left\{ v = a_1 u_1 + \cdots + a_n u_n : \begin{array}{l} u_1, \cdots, u_n \in V \\ a_1, \cdots, a_n \in F, \end{array} \; n \in \mathbb{N} = \{1, 2, \cdots\} \right\}$$

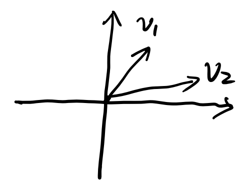
<u>Thm</u>:
- $\mathrm{Span}(S)$ is a subspace of V (proof by definition)
- Any subspace of V that contains S must also contain the $\mathrm{span}(S)$.
  (because subspace is closed under linear combinations)

In other words, $S \subseteq \underset{\substack{\uparrow \\ \text{subspace}}}{W} \subseteq V \implies \text{Span}(S) = W$.

Equivalently, $\text{span}(S) = \bigcap_{\substack{S \subseteq W \\ W \text{ subspace}}} W$ is the smallest subspace that contains S.

<u>Ex</u>: $\text{Span}\{v_1, v_2\} = \mathbb{R}^2$ if $v_2 \neq 0, v_1 \neq 0, v_1 \not\parallel v_2$ not parallel to

But if $\overset{\neq 0}{v_2} \parallel \overset{\neq 0}{v_1}$ (parallel), then

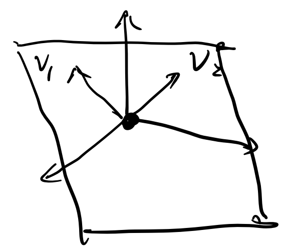


$\text{Span}\{v_1, v_2\} = \text{Span}\{v_1\} = \text{Span}\{v_2\}$ is a line passing through 0
parallel to v_1 origin.
representing the
0 vector.

<u>Ex</u>: $v_1, v_2 \in \mathbb{R}^3$, $v_1 \neq 0$, $v_2 \neq 0$, $v_1 \not\parallel v_2$

$\text{Span}\{v_1, v_2\}$ = plane containing v_1 and v_2 and passing through the origin

If $v_2 \parallel v_1$, then
$\text{Span}\{v_1, v_2\}$ is a line passing through 0.



<u>Ex</u>: Let $S = \{1, x, x^2\} \subset P(\mathbb{R})$
then $\text{Span}(S) = \{a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$
$= \{\text{polynomials of degree at most } 2\} = P_2(\mathbb{R})$

let $S' = \{1+x,\ x-x^2,\ x^2\}$

Then $\underline{\mathrm{Span}(S') = \mathrm{Span}(S) = P_2(\mathbb{R})}$.

Proof: Because any element of S' is a linear combination of vectors in S, we have $S' \subseteq \mathrm{Span}(S)$.

Because $\mathrm{span}(S)$ is a linear subspace and $\mathrm{Span}(S')$ is the smallest subspace containing S',

we have $\underline{\mathrm{Span}(S') \subseteq \mathrm{Span}(S)}$.

Similarly, we have $S \subseteq \mathrm{Span}(S')$:

$1 = 1+x-(x-x^2)-x^2$, $x = (x-x^2)+x^2$, $x^2 = x^2$.

So $\underline{\mathrm{Span}(S) \subseteq \mathrm{Span}(S')}$

So we get $\mathrm{span}(S') = \mathrm{span}(S) = P_2(\mathbb{R})$.

Ex: Is $\begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix}$ in $\text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}$?

$\begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix} \in \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}$ if and only if

there are $a, b \in \mathbb{R}$ s.t.

$$\begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + b \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

with Matrix Notation: $\begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$.

we use Gauss elimination to solve this system of linear equations.

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ -1 & 1 & -5 \\ 2 & 2 & -2 \end{array}\right) \xrightarrow[①\cdot(-2)+③]{①+②} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{②/2} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$②\cdot(-1)+①$

$\exists$ solution: $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \Leftarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right)$

so the answer is Yes.

Quiz:

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

- Define addition and scalar multiplication to make $M_{2\times 2}(\mathbb{R})$ become a vector space.

Sol: Addition: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$

Scalar multiplication: $\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \quad \forall \lambda \in \mathbb{R}.$

- Is $S_1 = \left\{ \underset{\overset{\parallel}{A}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in M_{2\times 2}(\mathbb{R}) : \underset{\overset{\parallel}{\mathrm{Tr}A}}{a+d} = 0 \right\}$ a subspace of $M_{2\times 2}(\mathbb{R})$

Sol: Yes. closed under addition, scalar multiplication (and $\overset{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\overset{\parallel}{0}} \in S_1$)

- Is $S_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) : ad - bc = 0 \right\}$ a subspace of $M_{2\times 2}(\mathbb{R})$

Sol: No. $\underset{\overset{\uparrow}{S_2}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + \underset{\overset{\uparrow}{S_2}}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin S_2$

Not closed under addition.