V : vector space over a field F — a set with vector addition and scalar multiplication which satisfies conditions (VS 1) - (VS 8).

Def: A subset $W \subseteq V$ is a subspace if W equipped with the vector addition and scalar multiplication from V becomes a vector space by itself.

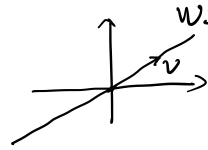
- $W \subseteq V$ is a subspace if the following 3 conditions are satisfied
  1. $\forall x, y \in W$, $x+y \in W$.
  2. $\forall x \in W$ $\forall a \in F$, $a \cdot x \in W$
  3. $0 \in W$

  These conditions also imply $-x = (-1) \cdot x \in W$, $\forall x \in W$. (In other words, the negative of any element in W is also contained in W)

Ex: trivial subspaces: zero subspace $\{0\}$ and the whole space V.

Ex: $V = \mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$.

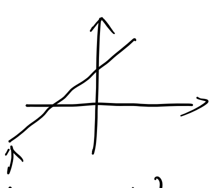
- possible subspaces:
  - $\{0\}$.
  - lines passing through $(0,0)$:



Any $v \neq 0$ generates a subspace $W = \{a \cdot v : a \in \mathbb{R}\}$

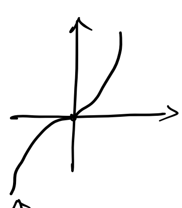
  - V

- Not subspaces:



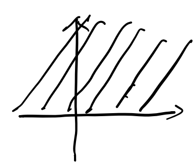
$S = \{x_1 + x_2 = 1\}$

$(0 \notin S)$



$x_2 = x_1^3$

$\left(\begin{array}{c} a \cdot v \notin W \\ \text{for } v \in W, a \neq 1 \end{array}\right)$



$S = \{(x_1, x_2) : x_2 \geq 0\}$

$(-v \notin W$ for $v \in W)$

Ex: $V = \mathbb{R}^3$

Possible subspaces: · $\{0\}$, V

· lines passing though $(0,0)$ (1-dim subspace)

· planes passing through $(0,0)$ (2-dim subspace)

Ex: $V = P(\mathbb{R}) = \{\underbrace{a_0 + a_1 t + \cdots + a_n t^n}_{= f} : a_i \in \mathbb{R}, i = 0, 1, \cdots, n\}$

$\deg(f) = \max\{i : a_i \neq 0\}$ Ex: $\deg(1+2t-t^3) = 3$.

$\deg(\underbrace{20}_{= 20 + 0\cdot t + 0 t^2}) = 0$.

Def $\deg(0) = -\infty$

Fix $d \in \{0, 1, 2, \cdots\} = \mathbb{Z}_{\geq 0}$

· $W_1 = P_d(\mathbb{R}) = \{f \in P(\mathbb{R}) : \deg(f) \leq d\}$
$= \{$polynomials of degree at most $d\}$

Then W_1 is a subspace. (it satisfies all 3 conditions for being a subspace)

· $S_1 = \{$polynomials of degree equal to $d\}$ is Not a subspace.

For example: when $d=1$, $t \in S_1$, $1-t \in S_1$, but
$t + (1-t) = 1 \notin S_1$. violating the 1st condition.

· $W_2 = \{f \in P(\mathbb{R}) : f(0) = a_0 = 0\}$ is a subspace.

Thm: Intersection of subspaces is a subspace:

If W_1, W_2 are subspaces of V, then $W_1 \cap W_2$ is also a subspace.

Pf: Condition 1: $\forall\, x, y \in W_1 \cap W_2,\ \ x+y \in W_1 \cap W_2$

2: $\forall\, x \in W_1 \cap W_2, \forall a \in F,\ \ ax \in W_1 \cap W_2$

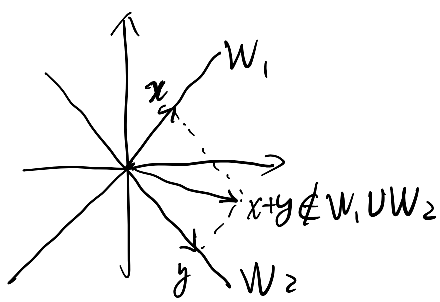
3: $0 \in W_1 \cap W_2$ ∎

However, in general union of subspaces is NOT a subspace:

$$W_1 \cup W_2 = \{x \in V : x \in W_1 \text{ or } x \in W_2\}$$

is in general not a subspace.

Ex:



$W_1 + W_2 = \mathbb{R}^2$.

The sum of two subspaces:

$W_1 + W_2 = \{x+y : x \in W_1, y \in W_2\}$ is a subspace:

Condition 1: $(x_1+y_1)+(x_2+y_2) = (x_1+x_2)+(y_1+y_2) \in W_1 + W_2$

2: $a(x+y) = ax + ay \in W_1 + W_2$

3. $0 = 0 + 0 \in W_1 + W_2$