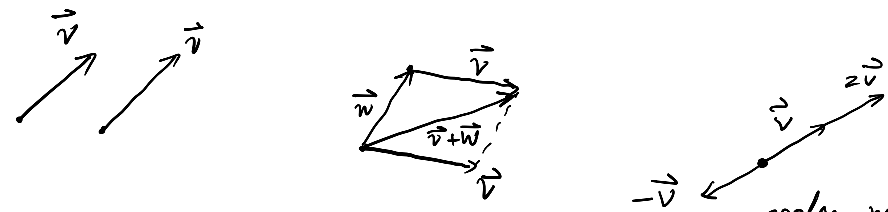
Ex: Vectors on $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$

$\uparrow$ real numbers.



vector addition

scalar multiplication.

$\rightsquigarrow$ Abstract generalization to vector space over a field F

- Def: A field F is a set with 2 operations $(+, \cdot)$ satisfying

$(F, +)$ satisfies:
- commutativity, associativity, existence of 0
- existence of negative $(a \mapsto -a)$.

$(F, \cdot)$ satisfies:
- commutativity, associativity, existence of $1 \in F$ $(a \cdot 1 = 1 \cdot a = a)$
- Existence of inverse for non-zero elements: $\forall a \neq 0 \in F, \exists b \in F$ s.t. $a \cdot b = b \cdot a = 1$

$(F, +, \cdot)$ satisfies distributive rule: $a \cdot (b + c) = a \cdot b + a \cdot c$.

Example of fields:
- $\mathbb{Q}$: field of rational numbers
- $\mathbb{R}$: - - - real numbers
- $\mathbb{C}$: - - - complex numbers.

- $\mathbb{Z} = \{\text{integers}\}$ is not a field: $\underset{\hat{\mathbb{Z}}}{2}$ does not have a inverse in $\mathbb{Z}$: $2^{-1} \notin \mathbb{Z}$.

$\mathbb{Z}$ is a (commutative) ring

$\uparrow$ does not require existence of inverse.

over a fixed field F

Def: A vector space V consists of a set together with two operations: vector addition and scalar multiplication such that the following conditions are satisfied:

(VS 1): $x+y=y+x$, $\forall x,y\in V$ (commutativity)

(VS 2): $(x+y)+z=x+(y+z)$, $\forall x,y,z\in V$ (associativity)

(VS 3): $\exists\, 0\in V$ s.t. $x+0=x$, $\forall x\in V$ (existence of 0)

(VS 4): $\forall x\in V$, $\exists\, y\in V$ s.t. $x+y=0$ (existence of $-x$ for all $x\in V$)

(VS 5): $1\cdot x=x$, $\forall x\in V$ ($1\in F$ is the identity)

(VS 6) $(ab)\cdot x=a\cdot(bx)$, $\forall a,b\in F$; $x\in V$.

(VS 7): $a(x+y)=ax+a\cdot y$, $\forall a\in F$; $x,y\in V$

(VS 8): $(a+b)x=ax+bx$, $\forall a,b\in F$; $x\in V$.

Examples: Ex 1: F a fixed field (e.g. $\mathbb{Q},\mathbb{R},\mathbb{C},\cdots$)

$$F^n=\{(a_1,\cdots,a_n):\ a_i\in F,\ i=1,\cdots,n\}$$

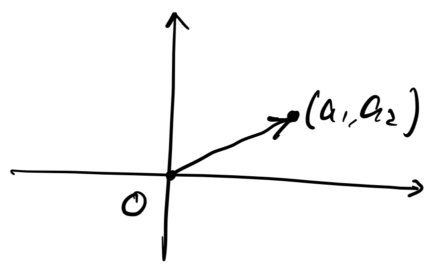
- $(a_1,\cdots,a_n)+(b_1,\cdots,b_n)=(a_1+b_1,\cdots,a_n+b_n)$
- $b\cdot(a_1,\cdots,a_n)=(ba_1,\cdots,ba_n)$

With these operations, F^n is a vector space over F.

In particular: $F = \mathbb{R}$, $n=2$: $\mathbb{R}^2 = \{(a_1,a_2) : a_1, a_2 \in \mathbb{R}\}$

$\parallel$

$\{$ vectors on $\mathbb{R}^2$ starting from $(0,0)$ and end at $(a_1,a_2)\}$



non
Ex $\quad F^2 = \{(a_1, a_2) : a_1, a_2 \in F\}$

- $(a_1,a_2) \oplus (b_1,b_2) = (a_1+b_1, a_2+2b_2)$ addition
- $b \cdot (a_1, a_2) = (ba_1, ba_2)$ scalar mult.

Under these operations, F^2 is Not a vector space:

commutativity is violated: (associativity also violated).

$$(b_1,b_2) \oplus (a_1,a_2) = (b_1+a_1,\ b_2+2a_2)$$

$$\neq$$

$$(a_1,a_2) \oplus (b_1,b_2) = (a_1+b_1,\ a_1+2b_2)$$

Ex: $V = \{\text{sequences of } F\}$

$= \{(a_1, a_2, \cdots) : a_i \in F, i = 1, 2, \cdots\}$

$= \{\text{functions from } \mathbb{N} \text{ to } F\}$

$\mathbb{N} = \{\text{natural numbers}\} = \{1, 2, 3, \cdots\}$

$$(a_1, a_2, a_3, \cdots) = \left\{ \begin{array}{l} 1 \mapsto a_1 \\ 2 \mapsto a_2 \\ 3 \mapsto a_3 \\ \cdots \end{array} \right\}$$

addition and scalar multiplication are defined similar to F^n (componentwise)

V is a vector space (of infinite dimension)

F^n - - - - - (of finite dimension)

Ex Polynomials with coefficients from F

$$P(F) = \left\{ a_0 + a_1 t + a_2 \cdot t^2 + \cdots + a_n \cdot t^n : \begin{array}{l} a_i \in F, \\ i = 1, \dots, n \end{array} \right\}$$

(dummy) variable

· addition (example): $(a_0 + a_1 \cdot t) + (b_0 + b_1 t + b_2 t^2 + b_3 t^3)$

$$= (a_0 + b_0) + (a_1 + b_1) t + b_2 t^2 + b_3 t^3$$

addition of coefficients of terms of same degree

· scalar multiplication: $b \cdot (a_0 + a_1 t + a_2 \cdot t^2 + \cdots + a_n \cdot t^n)$

$$= (b a_0) + (b a_1) \cdot t + (b a_2) t^2 + \cdots + (b a_n) \cdot t^n$$

$P(F)$ becomes a vector space over F.

- $\deg(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = \max\{ i : a_i \neq 0 \}$
- $P(F)$ can be embedded as a subspace of $\{\text{sequences}\}$

$$P(F) \longrightarrow \{\text{sequences}\}$$

$$a_0 + a_1 t + \cdots + a_n t^n \longmapsto (a_0, a_1, \dots, a_n, 0, 0, \dots)$$

sequence with finitely many non-zero components.

Thm (Cancellation law for vector spaces)

If $x+y = x+z$ for $x, y, z \in V$ ← a vector space

then $\qquad y = z$.

Proof: By (VS4), $\exists\, u \in V$ s.t. $x+u \underset{(VS\,1)}{=} u+x = 0$

$$
\begin{array}{ccccc}
x+y = x+z \implies & u+(x+y) & = & u+(x+z) \\
& \| \,(VS\,2) & & \| \\
& (u+x)+y & & (u+x)+z \\
& \| & & \| \\
& 0+y & & 0+z \\
& \| \,(VS\,3) & & \| \\
& y & = & z
\end{array}
$$

Cor: 0 from (VS 3) is unique: $\begin{array}{c} x+0 = x \\ \| \\ x+0' = x \end{array} \Rightarrow 0 = 0'$

Cor: $\forall x \in V$, $-x$ from (VS 4) is unique. $\begin{array}{c} x+y = 0 \\ \| \\ x+z = 0 \end{array} \Rightarrow \begin{array}{c} y = z \\ \| \\ -x \end{array}$