Quiz: Consider the linear system with a parameter a

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(i) When $a = 1$, find the general solution to the system. What type of equilibrium point do we have?

(ii) Find the values of a at which bifurcation happens. Describe how the type of equilibrium changes near the bifurcation value of a.

Solution: (i) First find eigenvalue and eigenvectors of $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (\lambda-1)^2+1=0 \Rightarrow \lambda-1=\sqrt{-1} \Rightarrow \lambda = 1 \pm i \qquad 1$$

$$\lambda = 1+i, \quad \begin{pmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{pmatrix} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \to \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \Rightarrow v = \begin{pmatrix} -i \\ 1 \end{pmatrix} \qquad 1$$

$$\Rightarrow Y_c(t) = e^{(1+i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix} = e^t(\cos t + i\sin t)\begin{pmatrix} -i \\ 1 \end{pmatrix} = e^t\begin{pmatrix} \sin t - i\cdot\cos t \\ \cos t + i\sin t \end{pmatrix}$$

$$= e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + i\cdot e^t\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} \qquad 1$$

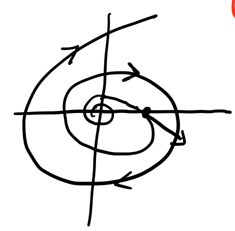
$\Rightarrow$ basic solutions $Y_1(t) = e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$, $Y_2(t) = e^t\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$ 1

$\Rightarrow$ general solution: $Y(t) = C_1 \cdot e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + C_2 \cdot e^t\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

$\mathrm{Re}\,\lambda = 1+i = 1 > 0 \Rightarrow$ spiraling source. 1

The vector field at $(1,0)$ is $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ spiraling clockwise



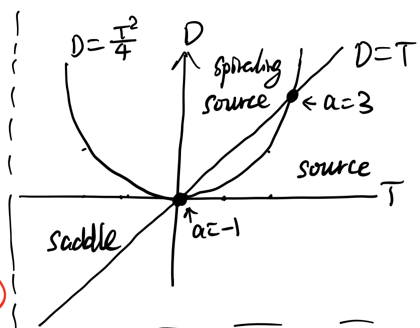
(ii) $\begin{pmatrix} a & 1 \\ -1 & 1 \end{pmatrix}$ trace $T = a+1$, determinant $D = a+1$

$\Rightarrow D = T = a+1$

bifurcation value:

$$\begin{cases} D = T = a+1 \\ D = \frac{T^2}{4} \end{cases} \Rightarrow T = \frac{T^2}{4} \Rightarrow T = 0 \text{ or } 4$$

$\Rightarrow a = -1$ or 3 2



$a < -1$: saddle point 1

$a = -1$: $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ repeated eigenvalue 0

$-1 < a < 3$ spiraling source 1

$a = 3$: $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ repeated eigenvalue 2

$a > 3$: source 1