Consider the differential equation:

$$y'' - 2y' + 5y = e^t \sin(2t) + t$$

Find the general solution.

Solution: • Associated homogeneous equation:

$$y'' - 2y' + 5y = 0$$

Characteristic polynomial: $\lambda^2 - 2\lambda + 5 = 0$

roots: $\lambda = \dfrac{2 \pm \sqrt{2^2 - 4\times 5}}{2} = 1 \pm \dfrac{\sqrt{-16}}{2} = 1 \pm 2i.$

$\Rightarrow$ basic solutions: $y_1(t) = e^t \cos(2t)$, $y_2(t) = e^t \sin(2t)$

$\Rightarrow$ general solution to the homogeneous equation:

$$y_h(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t) = C_1 \cdot e^t \cos(2t) + C_2 \cdot e^t \sin(2t).$$

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• Find a particular solution y_{p_1} to

$$y'' - 2y' + 5y = e^t \cdot \sin(2t)$$

complexify: $y'' - 2y' + 5y = e^t \cdot e^{2it} = e^{(1+2i)t}$

Since $1+2i$ is a root to the characteristic polynomial, we need to try $y_c = a \cdot t \cdot e^{(1+2i)t}$

$$\Rightarrow y_c' = a \cdot e^{(1+2i)t} + a(1+2i) \cdot t \cdot e^{(1+2i)t}$$

$$\Rightarrow y_c'' = 2a \cdot (1+2i) e^{(1+2i)t} + a \cdot (1+2i)^2 \cdot t \cdot e^{(1+2i)t}$$

$$\Rightarrow y_c'' - 2y_c' + 5y_c$$

$$= 2a \cdot (1+2i) e^{(1+2i)t} + \cancel{a \cdot (1+2i)^2 \cdot t \cdot e^{(1+2i)t}}$$

$$- 2 \cdot a \cdot e^{(1+2i)t} - \cancel{2a(1+2i) t \cdot e^{(1+2i)t}}$$

$$+ \cancel{5 \cdot a \cdot t \cdot e^{(1+2i)t}}$$

$$= 4a \cdot i \cdot e^{(1+2i)t} = e^{(1+2i)t}$$

$$\Rightarrow 4a \cdot i = 1 \Rightarrow a = \frac{1}{4i} = -\frac{1}{4} i$$

$$\Rightarrow y_c(t) = -\frac{1}{4} i \cdot t \cdot e^{(1+2i)t}$$

$$= -\frac{1}{4} i \cdot t \cdot e^{t}(\cos(2t) + i\sin(2t))$$

$\Rightarrow y_{p_1}$ = the imaginary part of $y_c(t)$

$$= -\frac{1}{4} t \cdot e^{t} \cos(2t).$$

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- Find a particular solution y_{p_2} to

$$y'' - 2y' + 5y = t.$$

Try $y_{p_2} = kt + b \Rightarrow y_{p_2}' = k, \; y_{p_2}'' = 0.$

$$\Rightarrow 0 - 2 \cdot k + 5 \cdot (kt + b) = 5kt + 5b - 2k = t$$

$$\Rightarrow \begin{cases} 5k = 1 \\ 5b - 2k = 0 \end{cases} \Rightarrow k = \frac{1}{5} \Rightarrow b = \frac{2k}{5} = \frac{2}{5} \times \frac{1}{5} = \frac{2}{25}$$

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$$\Rightarrow y_{p_2} = \frac{1}{5} t + \frac{2}{25}$$

- The general solution to $y'' - 2y' + 5y = e^t \sin(2t) + t$ is

$$y(t) = y_h + y_{p_1} + y_{p_2}$$

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$$= C_1 \cdot e^t \cos(2t) + C_2 e^t \sin(2t) - \frac{1}{4} t e^t \cos(2t) + \frac{1}{5} t + \frac{2}{25}$$