MAT 252 Fall 2021 MIDTERM II

NAME: **ID:**

THERE ARE FOUR (4) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		10pts
2		11pts
3		15pts
4		14pts
Total		50pts

!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(10pts) Solve the initial value problem:

$$y'' + y' - 2y = 3e^t, \quad y(0) = y'(0) = 1.$$

- associated homogeneous equation: $y'' + y' - 2y = 0$

characteristic polynomial: $\lambda^2 + \lambda - 2 = 0 = (\lambda+2)(\lambda-1)$

$\Rightarrow \lambda = -2 \Rightarrow y_1(t) = e^{-2t}$ $\Rightarrow$ general solution to the homogeneous eq.:

and $\lambda = 1 \Rightarrow y_2(t) = e^t$ $\quad y_h(t) = C_1 \cdot e^{-2t} + C_2 \cdot e^t$ 3

- Find a particular solution: because 1 is a root to the char. polynomial

we need to use $y_p(t) = a \cdot t\, e^t$

$\Rightarrow y_p' = a \cdot e^t + a \cdot t \cdot e^t, \quad y_p'' = 2a \cdot e^t + a \cdot t \cdot e^t$

$\Rightarrow y_p'' + y_p' - 2y_p = 2a \cdot e^t + a\cancel{t} \cdot e^t + a \cdot e^t + a\cancel{t}e^t - 2a\cancel{t}e^t$ 3

$= 3a \cdot e^t = 3 \cdot e^t \Rightarrow 3a = 3 \Rightarrow a = 1$

$\Rightarrow y_p(t) = t \cdot e^t$ 1

- general solution $y(t) = y_h(t) + y_p(t) = C_1 e^{-2t} + C_2 \cdot e^t + t \cdot e^t$

$y(0) = C_1 + C_2 = 1, \quad y'(0) = -2 \cdot C_1 + C_2 + 1 = 1 \Rightarrow C_2 = +2C_1$

$\Rightarrow C_1 + 2C_1 = 1 = 3C_1 \Rightarrow C_1 = \frac{1}{3}$ and $C_2 = 2C_1 = \frac{2}{3}$ 3

So $y(t) = \frac{1}{3} e^{-2t} + \frac{2}{3} e^t + t \cdot e^t$ is the unique solution to the initial value problem.

2(11pts) Consider a forced oscillation modeled on the equation:

$$y'' + 2y' + y = 4\sin(t).$$

(1) Find the general solution this equation.
(2) Find the steady state solution and its amplitude.

(1) • $y''+2y'+y=0$, $\lambda^2+2\lambda+1=(\lambda+1)^2=0 \Rightarrow \lambda_1=\lambda_2=-1=\lambda$

$\Rightarrow y_1(t)=e^{-t},\ y_2(t)=t\cdot e^{-t}$

$\Rightarrow y_h(t)=c_1\cdot e^{-t}+c_2\cdot t\cdot e^{-t}$ (4)

• complexify: $y''+2y'+y=4\cdot e^{it}=4(\cos t+i\cdot\sin t)$

Use $y_c(t)=a\cdot e^{it} \Rightarrow y_c'=a\cdot i\cdot e^{it},\ y_c''=-a\cdot e^{it}$ (3)

$\Rightarrow -a e^{it}+\underset{\underset{2a\cdot i\cdot e^{it}}{\parallel}}{2a\cdot i\cdot e^{it}}+a\cdot e^{it}=4\cdot e^{it} \Rightarrow 2a\cdot i=4 \Rightarrow a=\frac{4}{2i}=-2i$

$\Rightarrow y_c(t)=-2i\,e^{it}=-2i\cdot(\cos t+i\cdot\sin t)=+2\sin t-2i\cdot\cos t.$

$\Rightarrow y_p(t)=-2\cos t$ (2)

• $y(t)=y_h(t)+y_p(t)=c_1 e^{-t}+c_2\cdot t\cdot e^{-t}-2\cos t$ is the general solution

(2) The steady state solution is $y_s=y_p(t)=\underset{\underset{2\cos(t-\pi)}{\parallel}}{-2\cos t}.$ (2)

Its amplitude is 2.

3(15pts) Consider a Prey-Predator model

$$\begin{cases} \frac{dx}{dt} = -x + xy \\ \frac{dy}{dt} = y - xy \end{cases}$$

(1) Does x represent the prey (rabbit) or predator (fox)?
(2) Calculate the equilibrium solution.
(3) There is a conservation relation which can be expressed as an identity $F(x, y) =$ constant. Find the function F. (hint: divide the two equations and solve obtained differential equation $\frac{dy}{dx} = f(x, y)$).

(1) x represents the predator because the term $+xy$ shows that the encounter of x and y benefits x.

5

5

(2) $\begin{cases} -x+xy=0 = x\cdot(-1+y) \Rightarrow x=0 \text{ or } y=1 \\ y-xy=0 = y\cdot(1-x) \Rightarrow y=0 \text{ or } x=1 \end{cases} \Rightarrow \begin{cases} x=0 \text{ and } y=0 \\ \text{or} \\ x=1 \text{ and } y=1. \end{cases}$

$\Rightarrow$ the equilibrium solutions are $(x(t), y(t)) = (0,0)$ and $(x(t), y(t)) = (1,1)$.

(3) $\frac{dy}{dx} = \frac{y-xy}{-x+xy} = \frac{y(1-x)}{x(-1+y)} \Rightarrow \underset{\left(-\frac{1}{y}+1\right)dy}{\underset{\parallel}{\frac{-1+y}{y}dy}} = \underset{\left(\frac{1}{x}-1\right)dx}{\underset{\parallel}{\frac{1-x}{x}dx}}$

$\Rightarrow -\ln|y| + y = \ln|x| - x + C$

$\Rightarrow \ln|x| + \ln|y| - x - y = \text{Constant}$

5

$\Rightarrow F(x,y) = \ln|x| + \ln|y| - x - y$

Continuation of solution for problem **3**

(1)

4(14 pts) Find the general solution to the linear system:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \mathbf{Y}.$$

(2) Classify the equilibrium point (sink, source or saddle)?

- $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

$\Rightarrow \lambda_1 = 2, \ \lambda_2 = 3$ 3

- $\lambda_1 = 2$: $\begin{pmatrix} 1-2 & 1 \\ -2 & 4-2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 3

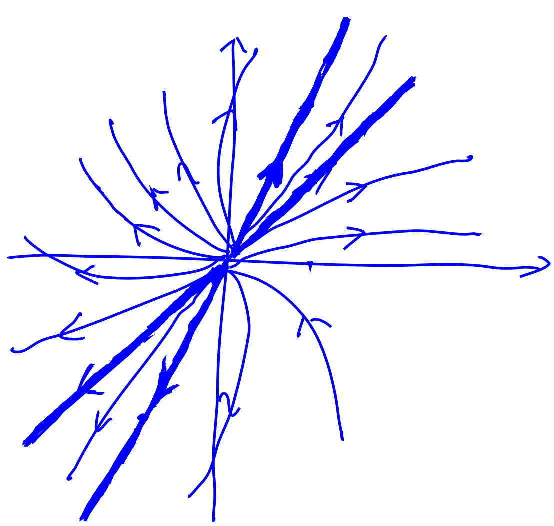
$\Rightarrow Y_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 3$: $\begin{pmatrix} 1-3 & 1 \\ -2 & 4-3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 3

$\Rightarrow Y_2(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow$ general solution : $Y(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 1

(2)



source 4

Continuation of solution for problem **4**

Scrap paper