MAT 252 Fall 2021 MIDTERM I

NAME: **ID:**

THERE ARE FOUR (4) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		10pts
2		10pts
3		15pts
4		15pts
Total		50pts

!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(10pts) Solve the initial value problem:

$$t\frac{dy}{dt} + ty = y, \quad y(-1) = 2.$$

$t\frac{dy}{dt} = y - ty = y(1-t)$ 1 $\Rightarrow y^{-1}\,dy = \frac{1-t}{t}\,dt = (t^{-1}-1)\,dt$ 2

$\Rightarrow \underbrace{\int y^{-1}\,dy}_{\ln|y|} = \underbrace{\int (t^{-1}-1)\,dt}_{\ln|t| - t + C_1}$ 2

$\Rightarrow |y| = e^{C_1}\cdot|t|\,e^{-t} \Rightarrow y = C\cdot t\cdot e^{-t}$ 2

$y(-1) = C\cdot(-1)\cdot e^{1} = -e\cdot C = 2 \Rightarrow C = -2\cdot e^{-1}$ 2

$\Rightarrow y(t) = -2\cdot e^{-1}\cdot t\cdot e^{-t} = -2\cdot t\cdot e^{-t-1}$ 1

Check: $y' = -2\cdot e^{-t-1} + 2\cdot t\cdot e^{-t-1}$

$ty' = -2\cdot t\cdot e^{-t-1} + 2t^2 e^{-t-1}$

$ty = -2\cdot t^2\cdot e^{-t-1}$

$ty' + ty = -2t\,e^{-t-1} = y$. $y(-1) = 2\cdot e^{+1-1} = 2$.

2(10pts) Use Euler's method to find numerical solution to the following equation on the interval $[0, 0.2]$ with step size 0.1. What value of $y(0.2)$ do you get? Don't round off your answer.

$$y' = t + y^2, \quad y(0) = 1.$$

- $y_1 = y_0 + f(t_0, y_0) \cdot \Delta t$ 5

  $= 1 + (0 + 1^2) \times 0.1 = 1 + 0.1 = 1.1$

- $y_2 = y_1 + f(t_1, y_1) \cdot \Delta t$

  $= 1.1 + (0.1 + (1.1)^2) \times 0.1$ 5

  $= 1.1 + (0.1 + 1.21) \times 0.1 = 1.1 + 0.131 = 1.231$

3(15pts) Assume that a tank contains 5 gallons of pure water at $t = 0$. At each minute, 2 gallons of salty water with a concentration 2 pound/gallon flows into the tank. At the same time, 3 gallons/min of well-mixed salty water in the tank flows out of the tank.

(1) Write down the initial value problem modeling the mixing process.

(2) What is the **concentration** of the salty water in the tank at $t = 4$ min?

y = amount of salt in the tank.

(1) $\frac{dy}{dt} = 2\times 2 - \frac{y}{5-t}\times 3 = 4 - \frac{3y}{5-t}$, $y(0) = 0$.

3

(2) $\frac{dy}{dt} + \frac{3}{5-t}y = 4$

integrating factor: $\mu(t) = e^{\int \frac{3}{5-t}dt} = e^{-3\cdot\ln(5-t)} = (5-t)^{-3}$

2

$\Rightarrow \frac{d}{dt}\left((5-t)^{-3}\cdot y\right) = 4\cdot(5-t)^{-3}$

$\Rightarrow (5-t)^{-3}\cdot y = \int 4\cdot(5-t)^{-3}dt = -4\cdot\frac{1}{-3+1}\cdot(5-t)^{-3+1} + C$

$= 2\cdot(5-t)^{-2} + C$

2

$\Rightarrow y(t) = 2\cdot(5-t) + C\cdot(5-t)^3$

2

$\Rightarrow y(0) = 10 + C\cdot 5^3 = 0 \Rightarrow C = -\frac{10}{5^3} = -\frac{2}{25}$

2

$\Rightarrow y(t) = 2\cdot(5-t) - \frac{2}{25}(5-t)^3$ amount of salt

2

$\Rightarrow$ concentration at $t = \frac{y(t)}{5-t} = 2 - \frac{2}{25}(5-t)^2$

2

$\Rightarrow$ at $t=4$, the concentration is $2 - \frac{2}{25}\times(5-4)^2 = \frac{2\times 24}{25} = \frac{48}{25}$ pound/gallon

$= 48\times 0.04$ pound/gallon.

$= 1.92$

2

Continuation of solution for problem **3**

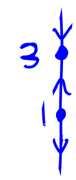
4(15 pts) Consider the population model:

$$\frac{dP}{dt} = 4P\left(1 - \frac{P}{4}\right) - h.$$

(1) Draw the phase diagram when $h = 3$.
(2) Assume $h = 3$ and the initial population is $P(0) = 2$. Sketch the graph of the solution. What is the limiting population as $t \to +\infty$?
(3) For what value of h do we have a bifurcation of the phase diagram? Explain your work.

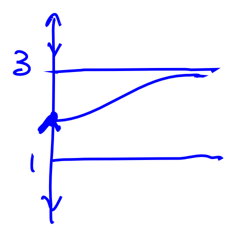
(1) $h=3$: $\frac{dP}{dt} = 4P(1-\frac{P}{4}) - 3 = -P^2+4P-3 = -(P-1)(P-3)$

phase line



4

(2)



3

$\lim_{t\to+\infty} P(t) = 3$.

2

(3) For any h, the equilibrium point of $\frac{dP}{dt} = 4P - P^2 - h$ is $= -(P^2-4P+h)$

$$\frac{4 \pm \sqrt{16-4h}}{2} = 2 \pm \sqrt{4-h}$$

1

For $h<4$, phase line  2 equilibrium pts

$h=4$, phase line  1 equilibrium pt

$h>4$. phase line  no equilibrium pts

3

2

the bifurcation happens at $h=4$ $\Leftarrow$

OR: the bifurcation value of h satisfies:

$$\begin{cases} 4P - P^2 - h = 0 \\ 4 - 2P = 0 \end{cases} \Rightarrow P=2 \text{ and } h = 4P - P^2 = 4\times 2 - 2^2 = 4$$

4

2

Continuation of solution for problem **4**

Scrap paper