MAT 252 Fall 2021 FINAL EXAM

NAME: **ID:**

THERE ARE SIX (6) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		16pts
2		16pts
3		16pts
4		16pts
5		16pts
6		20pts
Total		100pts

!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(16pts) Consider a logistic model with harvesting:

$$\frac{dP}{dt} = P(6-P) - h.$$

Assume that the initial population is $P(0) = 4$. Solve the following problems:

(**a**) If $h = 9$, find the value of P at $t = 4$.

(**b**) If $h = 5$, what is the population as $t \to +\infty$? Explain your conclusion.

Sol: (a) $\frac{dP}{dt} = P(6-P) - 9 = -P^2 + 6P - 9 = -(P-3)^2$ 1

$\Rightarrow -\frac{dP}{(P-3)^2} = dt$ 2 $\xRightarrow{\text{integrate}} -\frac{1}{-2+1}\cdot(P-3)^{-2+1} = t + C$ 2

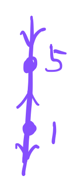
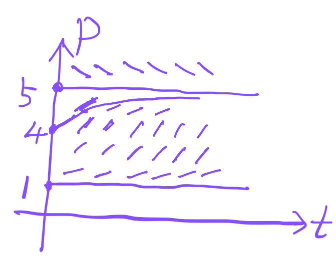
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{1}{P-3}$

when $t=0$, $\frac{1}{4-3} = 0 + C \Rightarrow C = 1 \Rightarrow \frac{1}{P-3} = t+1$ 1

$\Rightarrow P = \frac{1}{t+1} + 3 \Rightarrow P(4) = \frac{1}{5} + 3 = 3.2$ 2

(b) $\frac{dP}{dt} = P(6-P) - 5 = -(P^2 - 6P + 5) = -(P-1)(P-5)$ 2

phase line  directional field  3

$1 < P(0) < 5 \Rightarrow \lim_{t \to +\infty} P(t) = 5$ 3

2(16pts) Assume that a huge tank initially contains 4 gallons of salty water with a concentration 1/4 pound/gallon (lbs p/g). Assume that 2 gallons of salty water with a concentration 1/2 lbs p/g flows into the tank at each minute. At the same time, 1 gallon of well-mixed salty water flows out of the tank at each minute.

$$\left(\text{note: concentration} = \frac{\text{amount of salt}}{\text{volume of salty water}}\right)$$

(1) Write down and solve the initial value problem for the amount of salt in the tank.

(2) What is the **concentration** of the salty water in the tank at $t = 6$ min?

Sol: y = amount of salt

$$\frac{dy}{dt} = 2 \times \frac{1}{2} - \frac{y}{4+t} \times 1 = 1 - \frac{y}{4+t}, \quad y(0) = 4 \times \frac{1}{4} = 1.$$ 4

$$\Rightarrow \frac{dy}{dt} + \frac{1}{4+t} y = 1$$ integrating factor $\mu(t) = e^{\int \frac{1}{4+t} dt} = e^{\ln(4+t)}$
$= 4+t$ 4

$$\Rightarrow ((4+t)y)' = 4+t \Rightarrow (4+t) \cdot y = 4t + \frac{1}{2}t^2 + C$$ 2

Set $t=0$: $4 \times 1 = 0 + 0 + C \Rightarrow C = 4$ 2

$$\Rightarrow y = \frac{1}{4+t} \cdot \left(4t + \frac{1}{2}t^2 + 4\right)$$ 1

concentration at time $t = \frac{y(t)}{4+t} = \frac{1}{(4+t)^2}\left(4t + \frac{1}{2}t^2 + 4\right)$

3

at $t=6$: concentration $= \frac{1}{10^2} \times (24 + 18 + 4) = \frac{46}{100}$
$= 0.46$ ~~p/g~~ lbs/gl.

3(16pts) Consider the following equation for a forced oscillation :

$$\frac{d^2y}{dt^2} + y = 2\sin(t).$$

Solve the initial value problem with $y(0) = y'(0) = 1$. Is the amplitude of the oscillation bounded as $t \to +\infty$? What phenomenon do we have in this case?

associated homogeneous equation: $\frac{d^2y}{dt^2} + y = 0.$

characteristic polynomial: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$ 2

$\Rightarrow$ basic solutions $y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$. 2

Find a particular solution: complexity: $y'' + y = 2 \cdot e^{it}$

because the exponent i is a root to the char. polynomial.

try $y_c(t) = a \cdot t \cdot e^{it} \Rightarrow y_c' = a \cdot e^{it} + at \cdot i \cdot e^{it}$ 2

$y_c'' = a \cdot i \cdot e^{it} + a \cdot i \cdot e^{it} + a \cdot t \cdot (-1) \cdot e^{it}$

$\Rightarrow y_c'' + y_c = 2a \cdot i e^{it} = 2 \cdot e^{it} \Rightarrow 2ai = 2 \Rightarrow a = \frac{2}{2i} = -i$

$\Rightarrow y_c(t) = -i \cdot t \cdot e^{it} = -i \cdot t \cdot (\cos t + i \sin t) = \sin(t) - i \cdot t \cdot \cos t.$

$\Rightarrow$ the imaginary part $y_p(t) = -t \cdot \cos t$ is a particular solution 2

$\Rightarrow$ general solution to the non-hom. equation: 2

$y(t) = C_1 \cdot \cos t + C_2 \cdot \sin t - t \cdot \cos t.$

$(1-t)\cos t + 2 \cdot \sin t$

$\Rightarrow y(0) = C_1 = 1, \quad y'(0) = C_2 - 1 = 1 \Rightarrow C_2 = 2$ // 2

$\Rightarrow$ the solution to the initial value problem is $\boxed{\cos t + 2\sin t - t \cdot \cos t}$

The amplitude $\sqrt{(1-t)^2 + 4}$ is not bounded. 2 We have the $\boxed{\text{resonance}}$. 2

4(16 pts) Consider the linear system with a parameter:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ a & 0 \end{pmatrix} \mathbf{Y}.$$

(**a**) When $a = -1$, solve the linear system with initial value $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(**b**) For what values of a do we have bifurcations? Explain how the type of equilibrium point changes.

(a). $\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ Find eigenvalues / eigenvectors:

$\begin{vmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2+2\lambda+1 = (\lambda+1)^2 = 0 \Rightarrow \lambda = -1.$ repeated eigenvalue. 2

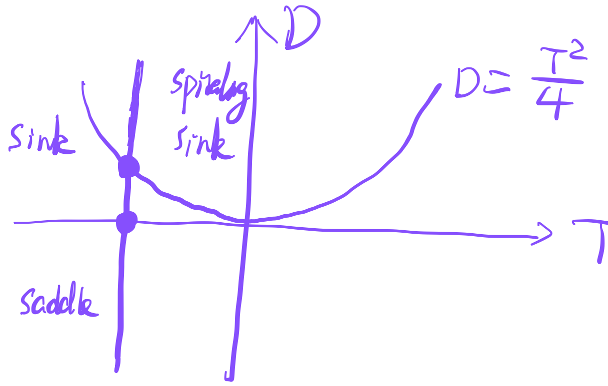
$\Rightarrow Y(t) = e^{\lambda t}\cdot\left(Y(0) + t\cdot(A-\lambda I)\cdot Y(0)\right)$ 3

$= e^{(-1)t}\cdot\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = e^{-t}\cdot\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$= e^{-t}\cdot\begin{pmatrix} 1+t \\ 2+t \end{pmatrix}.$ 3

(b) $\underbrace{\mathrm{tr}\begin{pmatrix} -2 & 1 \\ a & 0 \end{pmatrix} = -2}_{T}$, $\det\begin{pmatrix} -2 & 1 \\ a & 0 \end{pmatrix} = -a = D$ 1

bifurcation happens when $(-2, -a)$ lies on $\{D = 0\}$ or $\{D = \frac{T^2}{4}\}$



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$\Rightarrow \underset{\substack{\text{zero eigenvalue}\\ \downarrow}}{a = 0}$ or $-a = \frac{(-2)^2}{4} = 1 \Rightarrow a = -1$ ← repeated eigenvalue 2

$a < -1$: spiraling sink $\quad a > 0$: saddle point 3

$-1 < a < 0$: real sink

5(16pts) Solve the initial value problem:

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \mathbf{Y}(0) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

Find eigenvalues/eigenvectors:

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)\left((1-\lambda)(-1-\lambda)-3\right) = (2-\lambda)\cdot(\lambda^2-4)$$

$$= (2-\lambda)\cdot(\lambda-2)\cdot(\lambda+2)$$

$\Rightarrow \lambda = 2$ or $\lambda = -2$ 2

$\lambda = 2$: $\begin{pmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ 2 linearly independent eigenvectors:

free variables

$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ 4

$\Rightarrow Y_1(t) = e^{2t}\cdot v_1, \quad Y_2(t) = e^{2t} v_2$

$\lambda = -2$: $\begin{pmatrix} 3 & 0 & 3 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 2

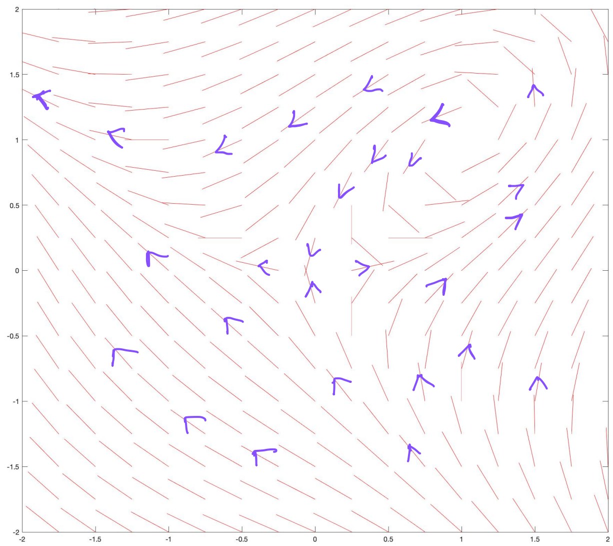
$\Rightarrow Y_3(t) = e^{-2t}\cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 2

$\Rightarrow$ general solution $Y(t) = c_1 \cdot e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \cdot e^{2t} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + c_3 \cdot e^{-2t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

set $t=0$: $Y(0) = c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + c_3 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ 2 3

$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3c_2 - c_3 \\ c_1 \\ c_2 + c_3 \end{pmatrix} \Rightarrow c_1 = 2, \quad c_2 = \frac{2+2}{4} = 1, \quad c_3 = 2 - c_2$

$$e^{2t}\begin{pmatrix}3\\2\\1\end{pmatrix}+e^{-2t}\begin{pmatrix}-1\\0\\1\end{pmatrix} = Y(t) = 2e^{2t}\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{2t}\begin{pmatrix}3\\0\\1\end{pmatrix} + e^{-2t}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$ 1



6(20pts) Consider the non-linear system:

$$\begin{cases} \frac{dx}{dt} = x - y^2 \\ \frac{dy}{dt} = -y + x^2. \end{cases}$$

(1) Find all equilibrium points. Calculate the linearization at each of them and classify the types of equilibrium points for the linearized systems.
(2) Is the above nonlinear system a Hamiltonian system? If yes, find a Hamiltonian function.
(3) What are the types for equilibrium points for the nonlinear system? Explain your reason.
(4) The above attached picture shows the directional field in the phase plane but misses arrow heads. Find the correct directions of the vectors and add arrow heads to the picture to show the correct flow directions of solution curves. Always explain your works.

(1) equilibrium points $\begin{cases} x - y^2 = 0 \\ -y + x^2 = 0 \end{cases}$ 1 $\Rightarrow x = y^2 = (x^2)^2 = x^4$

$\Rightarrow x^4 - x = x(x^3-1) = x(x-1)(x^2+x+1) = 0$

$\Rightarrow x = 0$ or $x = 1$

$\Rightarrow (x=0,\ y=0^2=0)$ or $(x=1,\ y=1^2=1)$

$\Rightarrow$ two equilibrium points $(0,0)$ and $(1,1)$. 2

$$\begin{cases} \dfrac{dx}{dt} = x - y^2 = f(x,y) \\ \dfrac{dy}{dt} = -y + x^2 = g(x,y) \end{cases}$$

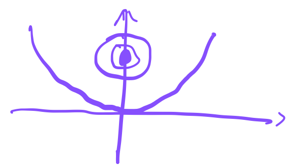
Continuation of Problem 6:

Jacobi matrix: $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -2y \\ 2x & -1 \end{pmatrix}$ 1

At $(0,0)$, $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$ saddle point 2

At $(1,1)$, $J(1,1) = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ $T = 0$. $D = -1 + 4 = 3 > 0$

2



center

(2) $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 1 + (-1) = 0 \Leftrightarrow \dfrac{\partial f}{\partial x} = -\dfrac{\partial g}{\partial y}$

$\Rightarrow$ the system is Hamiltonian

2

Find H: $\begin{cases} \dfrac{\partial H}{\partial y} = f = x - y^2 \Rightarrow H = \int (x - y^2)\,dy \\ \dfrac{\partial H}{\partial x} = -g = y - x^2 \end{cases}$ 1

$= xy - \frac{1}{3}y^3 + C(x)$

$\Rightarrow \dfrac{\partial H}{\partial x} = y + C' = y - x^2 \Rightarrow C' = -x^2 \Rightarrow C(x) = -\frac{1}{3}x^3$

1

$\Rightarrow H(x,y) = xy - \frac{1}{3}y^3 - \frac{1}{3}x^3$ is a Hamiltonian function.

1

Continuation of Problem 6:

(3) The linearized system at $(0,0)$ has a saddle point

$\Rightarrow$ The nonlinear system also has a saddle point at $(0,0)$.

The linearization at $(1,1)$ has a center. 2

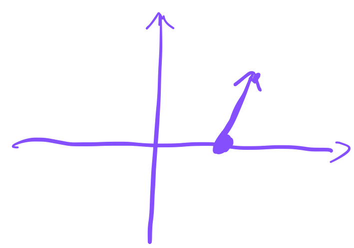
Because the nonlinear system is Hamiltonian, it can not have sinks or sources, so the nonlinear system also has a center at $(1,1)$ 2

(4) We can determine the rotational direction around the point $(1,1)$:

See the picture

$$J = J(1,1) = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}, \quad J\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Rightarrow$ rotating counterclockwise.



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One can also determine the direction of vector field by noting that near the saddle point $(0,0)$, the contracting (resp. expanding) direction is tangent to $(1,0)$ (resp. $(0,1)$).

OR: One can use the information from the x-nullcline, y-nullcline.

Extra page