Chains of generalized eigenvectors

Principle: For a **fixed** eigenvalue

- Number of Chains= Number of Eigenvectors=Multiplity-Defect;
- Sum of Length of Chains = Multiplicity of the Eigenvalue.

Caveat: Number of Eigenvectors, denoted by "# EVector" in the following charts, means the Number of **Linearly Independent** Eigenvectors.

- For 2×2 matrices, there are 3 possible cases:

  Case 1. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	1	1	$v_1 \to 0$	$e^{\lambda_1 t} v_1$
λ_2	1	1	$v_2 \to 0$	$e^{\lambda_2 t} v_2$

  Case 2. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	2	1	$v_1 \to v_2 \to 0$	$e^{\lambda_2 t}(v_1 + v_2 t)$, $e^{\lambda_1 t} v_2$

  Case 3. (Happens **only** for $A = \lambda_1 \mathbf{I}$)

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	2	2	$v_1 \to 0$, $v_2 \to 0$	$e^{\lambda_1 t} v_1, e^{\lambda_1 t} v_2$.

- For 3×3 matrices, there are 6 possible cases:

  Case 1. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	1	1	$v_1 \to 0$	$e^{\lambda_1 t} v_1$
λ_2	1	1	$v_2 \to 0$	$e^{\lambda_2 t} v_2$
λ_3	1	1	$v_3 \to 0$	$e^{\lambda_3 t} v_3$

  Case 2. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	1	1	$v_1 \to 0$	$e^{\lambda_1} v_1$
λ_2	2	1	$v_2 \to v_3 \to 0$	$e^{\lambda_2 t}(v_2 + v_3 t)$, $e^{\lambda_2 t} v_3$

Case 3. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	1	1	$v_1 \to 0$	$e^{\lambda_1 t} v_1$
λ_2	2	2	$v_2 \to 0$	$e^{\lambda_2 t} v_2$
			$v_3 \to 0$	$e^{\lambda_2 t} v_3$

Case 4. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	3	1	$v_1 \to v_2 \to v_3 \to 0$	$e^{\lambda_1 t}(v_1 + tv_2 + \frac{t^2}{2} v_3), e^{\lambda_1 t}(v_2 + tv_3), e^{\lambda_1 t} v_3$

Case 5. Example (Jordan form): $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	3	2	$v_1 \to 0$	$e^{\lambda_1 t} v_1$
			$v_2 \to v_3 \to 0$	$e^{\lambda_1 t}(v_2 + tv_3), e^{\lambda_1 t} v_3$

Case 6. (Happens **only** for $A = \lambda_1 \mathbf{I}$)

EValue	Mult.	# EVector	Chain	Basic Solution
λ_1	3	3	$v_1 \to 0,\ v_2 \to 0,\ v_3 \to 0$	$e^{\lambda_1 t} v_1, e^{\lambda_1 t} v_2, e^{\lambda_1 t} v_3.$

Lazier (rougher) way to write down basic solutions: For any fixed eigenvalue λ of multiplicity m. One can calculate the set of basic solutions as follows

1. Calculate $(A - \lambda I)^m$.
2. Find m **linearly independent** *generalized eigenvectors* $\{v_1, \cdots, v_m\}$. This means that:

$$(A - \lambda I)^m v_i = 0, \text{ for each } i = 1, \cdots, m.$$

3. Write down the m basic solutions for each v_i:

$$\begin{aligned} x_i(t) &= e^{\lambda t} \Big[v_i + t(A - \lambda I) v_i + \frac{t^2}{2!}(A - \lambda)^2 v_i + \cdots + \\ &\quad + \frac{t^{m-2}}{(m-2)!}(A - \lambda I)^{m-2} v_i + \frac{t^{m-1}}{(m-1)!}(A - \lambda I)^{m-1} v_i \Big]. \end{aligned}$$