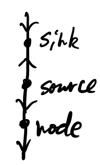
1、 Separable equations

$$\frac{dy}{dt} = f(t)\cdot g(y) \rightarrow \frac{dy}{g(y)} = f(t)dt$$

- Logistic model (with possible harvesting)

$$\frac{dP}{dt} = kP\cdot\left(1-\frac{P}{N}\right) - a$$

- autonomous equation . phase line




2. 1st linear equation

$$\frac{dy}{dt} + P(t)\, y = q(t).$$

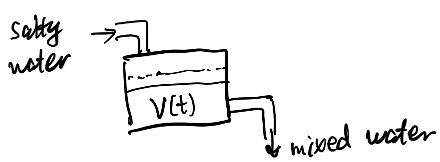
- integrating factor $\mu(t) = e^{\int P(t)dt}$

$$\Rightarrow \mu\cdot y' + \mu'\cdot y = (\mu\cdot y)' = q(t)\cdot\mu(t)$$

$$\Rightarrow \mu\cdot y = \int q(t)\cdot\mu(t) \Rightarrow y(t) = \frac{1}{\mu(t)}\cdot\int q(t)\cdot\mu(t)dt.$$

$$= \frac{1}{\mu(t)}\left(y_p(t) + C\right).$$

- Mixing Problem




$y$: amount of salt.

$$\frac{dy}{dt} = C_{in}\cdot V_{in} - \frac{y}{V(t)}\cdot V_{out}$$

$$\frac{dy}{dt} + \underbrace{\frac{V_{out}}{V(t)}}_{P(t)}\cdot y_{out} = C_{in}\cdot V_{in}$$

3. 2nd linear equations with constant coefficients.

① $y'' + p\cdot y' + q\cdot y = f(t)$.

- associated homogeneous equation $y'' + py' + qy = 0$.

characteristic polynomial: $\lambda^2 + p\lambda + q = 0 \Rightarrow \lambda = \frac{-p \pm \sqrt{p^2-4q}}{2}$

case 1: $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ $\quad y_1(t) = e^{\lambda_1 t}, \ y_2(t) = e^{\lambda_2 t}$

case 2: $\lambda_1 = \lambda_2 \in \mathbb{R}$ $\quad y_1(t) = e^{\lambda_1 t}, \ y_2(t) = t \cdot e^{\lambda_1 t}$

case 3: $\lambda_1 = a+bi \in \mathbb{R}$ $\quad y(t) = e^{(a+bi)t} = e^{at} \cdot (\cos bt + i \sin bt)$
$\lambda_2 = a-bi$ $\quad \Rightarrow y_1(t) = e^{at}\cos(bt), \ y_2(t) = e^{at}\sin(bt)$

general solution $y_h(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$.

- Find particular solutions to the non-homogeneous eq.

- $y'' + p \cdot y' + q \cdot y = e^{\mu \cdot t}$

if $\mu \neq \lambda_1$, or $\lambda_2$, $\quad y_p(t) = C \cdot e^{\mu t}$

if $\mu = \lambda_1$ or $\lambda_2$ $\quad y_p(t) = C \cdot t \cdot e^{\mu t}$

- $y'' + p \cdot y' + q y = e^{\mu t} \cos(b \cdot t)$. $\quad e^{\mu t} \cdot \sin(bt)$ imaginary

complexified equation $y'' + p y' + q y = e^{(\mu + ib)t}$
$= e^{\mu t}(\cos(bt) + i \sin(bt))$

if $\mu + ib \neq \lambda_1$ or $\lambda_2$, $y_c(t) = C e^{(\mu+ib)t}$

if $\mu + ib = \lambda_1$ or $\lambda_2$, $y_c(t) = C\, t\, e^{(\mu+ib)t}$
$= (r+is) \cdot t \cdot e^{\mu t} \cdot (\cos(bt) + i \sin(bt))$
$= t \cdot e^{\mu t} \cdot (r\cos(bt) - s \cdot \sin(bt) + i \cdot (r \sin(bt) + s \cdot \cos(bt)))$

$\Rightarrow y_p(t) = Re(y_c(t))$
or $Im(y_c(t))$

- $y'' + p y' + q y = t \cdot e^{\mu t} \cdot \cos(bt)$

$y_c(t) = (C_1 t + C_2) \cdot e^{(\mu + ib)t}$ if $\mu + ib \neq \lambda_1, \lambda_2$

$\uparrow t$ if $\mu + ib = \lambda_1$ or $\lambda_2$

General solution: $y = y_h + y_p$

$= C_1 \cdot y_1 + C_2 \cdot y_2 + y_p$.

- Oscillation

| | undamped | damped |
|---|---|---|
| un-forced | $b=0$ | |
| forced | $b=0$ | |

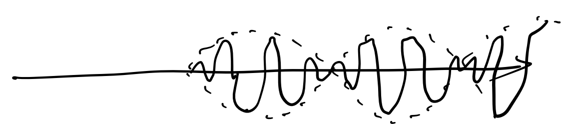
$$\underset{\text{mass}}{m}\, y'' + \underset{\text{damping}}{b}\, y' + \underset{\text{hook}}{k}\, y = \underset{\text{external force}}{f(t)}.$$

- undamped: $my'' + ky = \cos(\omega t)$.

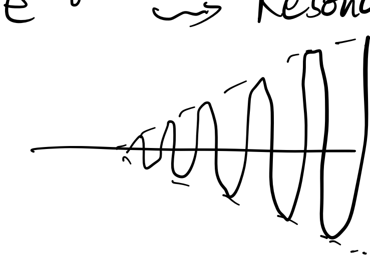
Sinusoidal

$\omega_0 = \sqrt{\frac{k}{m}}$ $\quad y_h = C_1 \cdot \cos(\omega_0 t) + C_2 \cdot \sin(\omega_0 t)$

- $\omega \neq \omega_0$ $\quad y_p(t) = c \cdot e^{i\omega t} \rightsquigarrow$ beats ($\omega \sim \omega_0$)



- $\omega = \omega_0$ $\quad y_c(t) = c\,t\,e^{i\omega_0 t} \rightsquigarrow$ Resonance



- damped case $\quad my'' + by' + ky = 0$ (unforced)

$\boxed{b>0}$

$$m\lambda^2 + b\lambda + k = 0 \Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

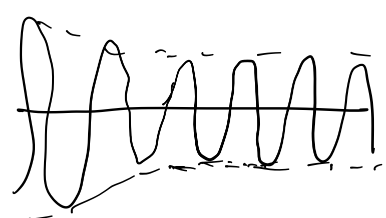
$Re(\lambda) < 0.$

$b^2 - 4mk > 0 \quad$ over-damped

$b^2 - 4mk < 0 \quad$ under-damped.

- $my'' + by' + ky = f(t)$
  - $e^{\mu t}$
  - $\boxed{\cos(\omega t)}$
  - $\sin(\omega t)$.

$y_p$ = steady state. $\quad (y_p = A\cos(\omega t) + B\sin(\omega t))$



$$y = y_h + y_p$$
$$= c_1 \cdot y_1 + c_2 \cdot y_2 + y_p$$

$y_1 = e^{\lambda_1 t} \qquad y_2 = e^{\lambda_2 t}$

$e^{Re(\lambda_1)t} \cdot \cos(Im(\lambda_1)t) \qquad e^{Re(\lambda_1 t)} \sin(Im(\lambda_1)t)$

$\downarrow t \to +\infty \qquad \downarrow t \to +\infty$

$0 \qquad 0$

# 4. linear system

$$\frac{dY}{dt} = A \cdot Y \qquad A: 2\times 2 \text{ matrix.}$$

- Find eigenvalues/eigenvector $|A-\lambda I|$

  - $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ $\quad Y_1(t) = e^{\lambda_1 t} v_1, \quad Y_2(t) = e^{\lambda_2 t} v_2$

    $\lambda_1 \rightsquigarrow v_1, \quad \lambda_2 \rightsquigarrow v_2$

  - $\lambda_1 = \lambda_2 = \lambda$

    $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\quad Y_1(t) = e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Y_2(t) = e^{\lambda t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

    <u>only 1 linearly independent eigenvector</u>

    $v_0 \to v_1 \to 0 \qquad Y_1(t) = e^{\lambda t} v_1$

    $(A-\lambda I) v_0 = v_1 \qquad \underline{Y_2(t) = e^{\lambda t}(v_0 + v_1 t)}$

    $\Rightarrow Y(t) = c_1 Y_1 + c_2 \cdot Y_2$

    $Y(t) = e^{\lambda t} \cdot (u_0 + (A-\lambda I) u_0 \cdot t)$, $\quad u_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

- $\lambda_1 = a + ib \quad b \neq 0$
  $\lambda_2 = a - ib$
  $\rightsquigarrow v_c = u_0 + i u_1$

  $\rightsquigarrow Y_c(t) = e^{(a+ib)t}(u_0 + i u_1) = e^{at}(\cos bt + i \sin bt) \cdot (u_0 + i u_1)$

  $= e^{at} \cdot [\underline{\cos(bt) \cdot u_0 - (\sin bt) u_1} + i \cdot (\cos(bt) u_1 + \sin(bt) u_0)$

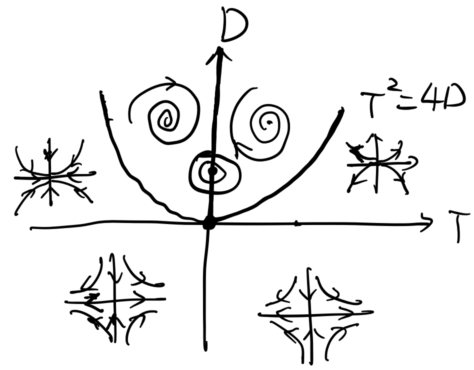
  $\rightsquigarrow Y_1(t) = e^{at}(\cos(bt) u_0 - \sin(bt) u_1)$

  $Y_2(t) = e^{at} \cdot (\cos(bt) u_1 - \sin(bt) u_0)$

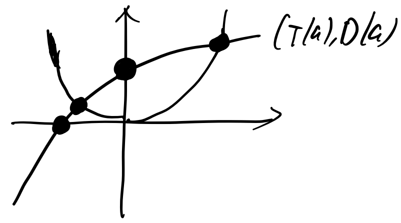
# Trace-Determinant Plane

$$\begin{pmatrix} T = tr(A) = \lambda_1+\lambda_2 \\ D = det(A) = \lambda_1\lambda_2 \end{pmatrix} \quad \lambda^2 - T\lambda + D = 0$$

$$\lambda = \frac{T \pm \sqrt{\boxed{T^2-4D}}}{2}$$




$$A = A(a) \rightsquigarrow \begin{matrix} T = T(a) \\ D = D(a) \end{matrix}$$




Refined information:
- straight line solution $\rightsquigarrow$ direction of eigenvectors
- spiraling direction $\rightsquigarrow$ direction of vector field at $(1,0)$

3x3: $\frac{dY}{dt} = A \cdot Y$. $\quad |A-\lambda I| = 0 \rightsquigarrow \lambda_1, \lambda_2, \lambda_3$

- $\lambda_1 \to v_1$, $\lambda_2 \to v_2$, $\lambda_3 \to v_3$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ $\quad Y_i(t) = e^{\lambda_i t} v_i, \quad i = 1, 2, 3.$

- $\lambda_1$ real, $\lambda_2 = a+ib \rightsquigarrow v_c$, $\lambda_3 = a-ib$; $\lambda_1 \to v_1$
  $Y_1(t) = e^{\lambda_1 t} v_1$
  $Y_2(t) = Re\left(e^{(a+ib)t} v_c\right)$
  $Y_3(t) = Im\left(e^{(a+ib)t} v_c\right)$

- $\lambda_1 = \lambda_2 \neq \lambda_3$, $\lambda_3 \to v_3$ $\quad e^{\lambda_3 t} v_3$
  - 2 eigenvectors $v_1, v_2$ associated to $\lambda_1 = \lambda_2 \Rightarrow Y_1(t) = e^{\lambda_1 t} v_1$, $Y_2(t) = e^{\lambda_2 t} v_2$
  - 1 eigenvector $u_0 \to u_1 \to 0$ $\quad Y_1(t) = e^{\lambda t} u_1$
    $(A - \lambda I) u_0 = u_1 \quad Y_2(t) = e^{\lambda t}(u_0 + u_1 t).$

$\lambda_1=\lambda_2=\lambda_3$

5. Nonlinear system

$$\begin{cases} \dfrac{dx}{dt}=f(x,y)\\ \dfrac{dy}{dt}=g(x,y).\end{cases}$$

- Find equilibrium points $\begin{cases} f(x_0,y_0)=0\\ g(x_0,y_0)=0.\end{cases}$

- Calculate the linearization at $(x_0,y_0)$

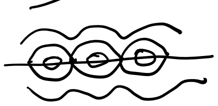
$$J(x,y)=\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix} \rightsquigarrow \frac{d}{dt}Y=J(x_0,y_0)\cdot Y.$$

classify type of equilibrium points.

- Sketch the diagram ( x-nullcline, y-nullcline)

- Hamiltonian system $\begin{cases}\dfrac{dx}{dt}=\dfrac{\partial H}{\partial y}\\ \dfrac{dy}{dt}=-\dfrac{\partial H}{\partial x}\end{cases}$ determine Hamiltonian or not find H if yes.

Pendulum



Gradient system

Fact: Hamiltonian system can not have sink or source because of the conservation of H (the energy.)