$\frac{dY}{dt} = A \cdot Y \qquad Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \qquad A$: $n \times n$ matrix

<u>Fact</u>: There exist n <u>linearly independent</u> basic solutions $Y_1, \cdots, Y_n$.

($Y_1(0), \cdots, Y_n(0)$ are linearly independent)

$\Rightarrow$ The general solution $Y(t) = C_1 \cdot Y_1(t) + \cdots + C_n Y_n(t)$

To find $Y_1, \cdots, Y_n$, need to find eigenvalues/eigenvectors of A

$n=2$ case 1: $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ $\Rightarrow$ $Y_1(t) = e^{\lambda_1 t} v_1$

$\lambda_1 \rightsquigarrow v_1$, $\lambda_2 \rightsquigarrow v_2$ $\qquad Y_2(t) = e^{\lambda_2 t} v_2$.

case 2: $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$

<u>case 2a</u>: $A = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \Rightarrow Y_1(t) = e^{\lambda t} \underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$, $Y_2(t) = e^{\lambda t} \underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

2 linearly indep. eig. vectors

<u>case 2b</u> $\qquad Y_1(t) = e^{\lambda t} v_1$

$\boxed{\text{1 linearly ind. eig. vector.}}$ $\qquad \boxed{Y_2(t) = e^{\lambda t} \cdot (v_0 + v_1 t) \quad v_0 \to v_1 \to 0.}$

<u>case 3</u>: $\lambda_1 = a+bi, \lambda_2 = a-bi$. $\qquad e^{\lambda t} v_1 = \underline{Y_1(t)} + i \underline{Y_2(t)}$

$\lambda_1 \rightsquigarrow v_1 \qquad e^{\lambda t} v_1 = Y_c(t)$

<u>$n=3$</u>: A: 3×3 matrix

<u>case 1</u>: $\lambda_1 \neq \lambda_2 \neq \lambda_3$, $\lambda_1 \neq \lambda_3$ $\in \mathbb{R}$

3 real eigenvalues

$\lambda_1 \rightsquigarrow v_1 \rightsquigarrow Y_1(t) = e^{\lambda_1 t} v_1$

$\lambda_2 \rightsquigarrow v_2 \rightsquigarrow Y_2(t) = e^{\lambda_2 t} v_2$

$\lambda_3 \rightsquigarrow v_3 \rightsquigarrow \underline{Y_3(t) = e^{\lambda_3 t} v_3}$.

Straight line solutions.

$\lambda_1, \lambda_2, \lambda_3 > 0$: source

$\lambda_1, \lambda_2, \lambda_3 < 0$: sink

$\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 > 0$ saddle

$\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 > 0$ saddle

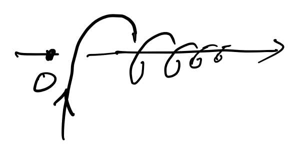
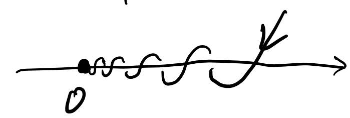
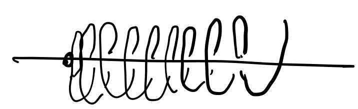
case 2: 1 real eigenvalue and 2 complex eigenvalues

$\lambda_1 \rightsquigarrow v_1$

$\lambda_2 = a + bi \rightsquigarrow v_2$

$\lambda_3 = a - bi \rightsquigarrow \overline{v_2}$

$\rightsquigarrow Y_1(t) = e^{\lambda_1 t} v_1$, $\quad Y_c(t) = e^{(a+bi)t} v_2 = Y_2(t) + i Y_3(t)$

- $\lambda_1 > 0$, $Re(\lambda_2) = a > 0$ source
- $\lambda_1 > 0$, $Re(\lambda_2) = a < 0$ saddle 
- $\lambda_1 < 0$, $Re(\lambda_2) < 0$ sink 
- $\lambda_1 < 0$, $Re(\lambda_2) > 0$ saddle
- $\lambda_1 > 0$, $Re(\lambda_2) = 0$ source 

Ex: $A = \left(\begin{array}{cc|c} 2 & -3 & 0 \\ -3 & 2 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$, $\begin{cases} \frac{dx}{dt} = 2x - 3y \\ \frac{dy}{dt} = -3x + 2y \\ \frac{dz}{dt} = -z \end{cases}$

· Find eigenvalues / eigenvectors.

$$A - \lambda I = \begin{vmatrix} +2-\lambda & -3 & 0 \\ -3 & +2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (\lambda^2 - 4\lambda + 4 - 9)\cdot(-\lambda - 1)$$

$$= -(\lambda^2 - 4\lambda - 5)\cdot(\lambda+1)$$

$$= -(\lambda-5)\cdot(\lambda+1)(\lambda+1) = -(\lambda-5)\cdot\underline{(\lambda+1)^2} = 0$$

$\Rightarrow \lambda = 5, -1, -1$ saddle point

· $\lambda_1 = 5$ $\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

free

$\rightsquigarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = v_1 \Rightarrow \boxed{Y_1(t) = e^{5t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$

· $\underline{\lambda = -1}$: $\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x - y = 0$

$\boxed{\text{free free}}$

$$\Rightarrow \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{Y_2(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \quad \boxed{Y_3(t) = e^{-t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}.$$

Ex: $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

multiplicity

$$-(5-\lambda)(\lambda+1)^{2}$$

$$\parallel$$

- $|A-\lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (5-\lambda)\cdot(-1-\lambda)(-1-\lambda).$

$\Rightarrow \lambda_1 = 5, \ \lambda_2 = -1, \ \lambda_3 = -1.$

- $\lambda_1 = 5 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v_1$

free   leading

$$\Rightarrow \boxed{Y_1(t) = e^{5t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$\boxed{\cdot\ \lambda = -1}$

$\underline{\text{multiplicity } 2}$

$$\underbrace{\overset{A-\lambda I}{\overset{\|}{\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}}} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \underset{\text{free}}{0} & 0 \end{pmatrix}$$

$$\Rightarrow \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \boxed{Y_2(t) = e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}.$$

$A-(-I)$

$\|$

$$\boxed{u_0 \xrightarrow{A+I} \underset{\substack{\| \\ v_2}}{u_1} \xrightarrow{A+I} 0}$$

eigenvector

solve $(A+I)u_0 = v_2$

$$\left(\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & \underset{\text{free}}{0} & 0 & 0 \end{array}\right)$$

$x = 0$

$z = 1$

$$\Rightarrow \quad u_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{Y_3(t) = e^{-t} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]}.$$

Ex: $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

$\boxed{\text{Jordan form}}$

- $|A-\lambda I| = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)^{3} = -(\lambda+1)^3$

$\Rightarrow \boxed{\lambda = -1 \quad \text{multiplicity } 3}$

- $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\boxed{\text{free}}$ $\Rightarrow \boxed{Y_1(t) = e^{-t} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$\boxed{\text{Chain of length 3}}$: $u_0 \xrightarrow{A+I} u_1 \xrightarrow{A+I} u_2 \xrightarrow{A+I} 0$

eigenvector $= v_1$

$(A+I)u_1 = v_1$, $(A+I)u_0 = u_1$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow u_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$y=0$
$z=1$

$$\Rightarrow \boxed{\begin{aligned} Y_2(t) &= e^{-t}\cdot\left(u_1 + t\cdot u_2\right) \\ Y_3(t) &= e^{-t}\cdot\left(u_0 + t u_1 + \frac{t^2}{2} u_2\right) \end{aligned}}$$

$$Y(t) = e^{\lambda t}\cdot\left[u_0 + t\cdot(A-\lambda I)u_0 + \frac{t^2}{2}\cdot(A-\lambda I)^2 u_0\right]$$

solution to the IVP $\begin{cases} \frac{dY}{dt} = A\cdot Y \\ Y(0) = u_0 \end{cases}$

if λ is an eigenvalue of multiplicity 3

Lorenz Equations

$$
\begin{cases}
\dfrac{dx}{dt} = 10(y-x) = f \\[2mm]
\dfrac{dy}{dt} = 28x - y - \underline{xz} = g \\[2mm]
\dfrac{dz}{dt} = -\dfrac{8}{3}z + \underline{xy} = h
\end{cases}
$$

Equilibrium points

$$
\begin{cases}
y - x = 0 \Rightarrow x = y \\
28x - y - xz = 0 \Rightarrow 27x - xz = 0 \Rightarrow \underline{x=0 \text{ or } z=27} \\
-\frac{8}{3}z + xy = 0 \Rightarrow \underline{x^2 = \frac{8}{3}z}
\end{cases}
$$

$\underline{x = 0 = y,\ z = 0.}$

$\boxed{(0,0,0)}$

$x = y = \sqrt{\frac{8}{3} \times 27} = \sqrt{8 \times 9} = \pm 3 \times 2\sqrt{2} = \pm 6\sqrt{2}$

$\boxed{(6\sqrt{2}, 6\sqrt{2}, 27)}$, $\underline{(-6\sqrt{2}, -6\sqrt{2}, 27)}$

$\underline{\text{Linearize}}$:

$$
\boxed{J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial y} \end{pmatrix}} = \begin{pmatrix} -10 & 10 & 0 \\ 28-z & -1 & -x \\ y & x & -\frac{8}{3} \end{pmatrix}
$$

$$J(0,0,0) = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}$$

$$\begin{vmatrix} -10-\lambda & 10 \\ 28 & -1-\lambda \end{vmatrix} = \lambda^2 + 11\lambda + \underbrace{10-280}_{\substack{\parallel \\ -270<0 \\ \parallel \\ D}} \Rightarrow \begin{array}{l} \lambda_1 < 0 < \lambda_2 \\ \lambda_3 = -\frac{8}{3} < 0 \end{array}$$

$\Rightarrow$ $(0,0,0)$ is a saddle pt.

$$J(6\sqrt{2}, 6\sqrt{2}, 27) = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -6\sqrt{2} \\ 6\sqrt{2} & 6\sqrt{2} & -\frac{8}{3} \end{pmatrix}$$

$$\Rightarrow \quad \lambda_1 \approx -13.8, \quad \lambda_2 \approx 0.094 + 10.22i$$
$$\lambda_3 \approx 0.094 - 10.22i$$

$\Rightarrow$ saddle point