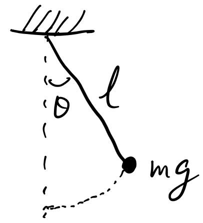
Nonlinear Pendulum



$$\boxed{m l \cdot \frac{d^2\theta}{dt^2} + b\cdot l \frac{d}{dt}\theta + m\cdot g\cdot \sin\theta = 0.}$$

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0. \qquad v = \frac{d\theta}{dt}$$

$$\Updownarrow$$

$$\begin{cases} \frac{d\theta}{dt} = v \\ \frac{dv}{dt} = \frac{d^2\theta}{dt^2} = -\frac{b}{m}\underbrace{\frac{d\theta}{dt}}_{= v} - \frac{g}{l}\sin\theta \end{cases} \iff \begin{cases} \frac{d\theta}{dt} = v \\ \frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{l}\sin\theta \end{cases}$$

- If $b=0$:

$$\begin{cases} \frac{d\theta}{dt} = v & = f(\theta, v) \\ \frac{dv}{dt} = -\frac{g}{l}\sin\theta = h(\theta, v) \end{cases} \qquad \frac{\partial f}{\partial \theta} = 0 = -\frac{\partial h}{\partial v} \Rightarrow \text{Hamiltonian system}$$

There exists H s.t.

Hamiltonian function: $$\boxed{H = \frac{1}{2}v^2 - \frac{g}{l}\cos\theta} = H(\theta, v). \Leftarrow \begin{cases} \frac{\partial H}{\partial v} = v \\ \frac{\partial H}{\partial \theta} = \frac{g}{l}\sin\theta \end{cases}$$

$$(ml^2)\cdot H = \underbrace{\frac{1}{2}m(l\dot\theta)^2}_{\text{kinetic}} - \underbrace{mg\cdot l\cos\theta}_{\text{potential}} \quad \text{energy} \Rightarrow \frac{d}{dt}H(\theta(t), v(t)) = 0.$$

$$\Rightarrow H(\theta(t), v(t)) = \text{constant}$$

- If $\boxed{b>0}$,

$$\begin{cases} \frac{d\theta}{dt} = v \\ \frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{l}\sin\theta. \end{cases} \qquad L(\theta, v) = \frac{1}{2}v^2 - \frac{g}{l}\cos\theta. \;\; (= H(\theta, v))$$

$$\frac{dH}{dt} = \frac{d}{dt}H(\theta(t), v(t)) = \frac{\partial H}{\partial \theta}\cdot\frac{d\theta}{dt} + \frac{\partial H}{\partial v}\frac{dv}{dt}$$

$$= \cancel{\frac{g}{l}\sin\theta\cdot v} + v\cdot\left(-\frac{b}{m}v - \cancel{\frac{g}{l}\sin\theta}\right) = \underline{-\frac{b}{m}v^2 \le 0}.$$

$\Rightarrow H(\theta(t), v(t))$ decreases along any solution $(\theta(t), v(t))$

Say H is <u>Lyapunov function</u> for the system.

Def: $L(x,y)$ is a Lyapunov function for a system of eqns. if for every solution $(x(t), y(t))$ that is not an equilibrium solution we have $\frac{d}{dt} L(x(t), y(t)) \leq 0$ $(<)$ for all t with strictly inequality except for a discrete set of t.

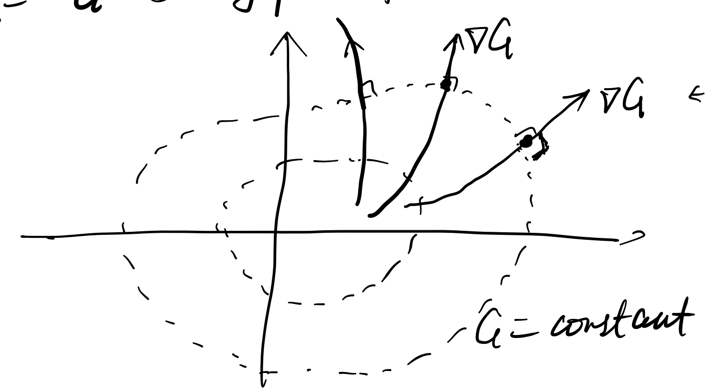
- Gradient system: if there is a function G s.t. ← potential function.

$$\begin{cases} \frac{dx}{dt} = \frac{\partial G}{\partial x} \\ \frac{dy}{dt} = \frac{\partial G}{\partial y} \end{cases} \qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial G}{\partial x} \\ \frac{\partial G}{\partial y} \end{pmatrix} = \nabla G$$

gradient of G.

$$\frac{d}{dt} G(x(t), y(t)) = \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt}$$
$$= \frac{\partial G}{\partial x}\cdot\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial G}{\partial y} \geq 0.$$

$L = -G$ is Lyapunov function: $\frac{d}{dt} L = -\frac{d}{dt} G \leq 0$.



$\frac{\nabla G}{|\nabla G|}$ is the direction along which G is increasing most rapidly.

(x_0, y_0) is equilibrium point.

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial G}{\partial x} & \frac{\partial}{\partial y}\frac{\partial G}{\partial x} \\ \frac{\partial}{\partial x}\frac{\partial G}{\partial y} & \frac{\partial}{\partial y}\frac{\partial G}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 G}{\partial x^2} & \frac{\partial^2 G}{\partial x \partial y} \\ \frac{\partial^2 G}{\partial x \partial y} & \frac{\partial^2 G}{\partial y^2} \end{pmatrix}$$

$||$

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda^2 - (\alpha+\gamma)\lambda + \alpha\gamma - \beta^2 = 0$$

$$\Rightarrow \lambda = \frac{(\alpha+\gamma) \pm \sqrt{(\alpha+\gamma)^2 - 4(\alpha\gamma - \beta^2)}}{2}$$

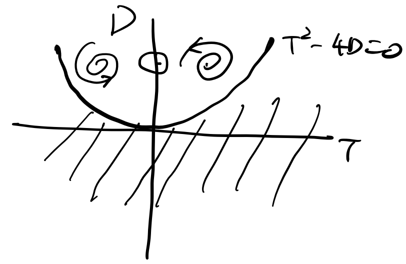
$$\alpha^2 + \gamma^2 + \overbrace{2\alpha\gamma - 4\alpha\gamma}^{-2\alpha\gamma} + 4\beta^2$$

$$= (\alpha-\gamma)^2 + 4\beta^2 \geq 0$$

$$= T^2 - 4D \geq 0$$

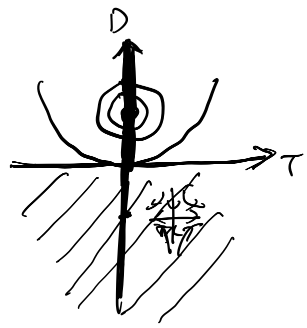
$\Rightarrow \lambda_{1,2}$ are real numbers.

$\Rightarrow$ The equilibrium point (x_0, y_0) can not be spiraling sink/source center.



For Hamiltonian system, linearization $f = \frac{\partial H}{\partial y}$, $g = -\frac{\partial H}{\partial x}$

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \\ -\frac{\partial^2 g}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$



$T = 0$,

$D = -\alpha^2 - \beta\gamma$