Hamiltonian System : there is a function $H = H(x,y)$, s.t.

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial y} = f \\ \dfrac{dy}{dt} = -\underline{\dfrac{\partial H}{\partial x}} = g \end{cases}$$

$\uparrow$ Hamiltonian function

$\Rightarrow$ · H is a conserved quantity : if $(x(t), y(t))$ is a solution, then:

$$\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0$$

$\Rightarrow \boxed{H(x(t), y(t)) = \text{constant}}$.

· linearization at an equilibrium point $\underline{(x_0, y_0)}$

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial y \partial x} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\partial H}{\partial x}\right) = -\frac{\partial^2 H}{\partial x^2}$, $\quad \frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = -\frac{\partial^2 H}{\partial y \partial x}$

$T = 0$, $D = -\alpha^2 - \beta\gamma$

$\lambda^2 - \lambda \cdot T + D = 0$

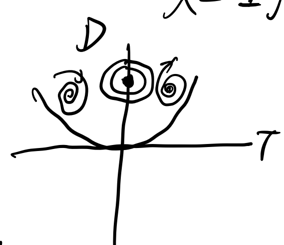
$\| $

$\lambda^2 + D = 0 \quad \Leftrightarrow \lambda^2 = -D = \alpha^2 + \beta\gamma$

$\Downarrow$

$\lambda = \pm\sqrt{\alpha^2 + \beta\gamma}$.

- $\alpha^2 + \beta\gamma > 0$. $\lambda_1 = \sqrt{\alpha^2 + \beta\gamma} = -\lambda_2$ $\boxed{\text{saddle}}$
- $\alpha^2 + \beta\gamma = 0$. $\underline{\lambda_1 = \lambda_2 = 0}$. degenerate
- $\underset{= -D}{\alpha^2 + \beta\gamma} < 0$. $\lambda_1 = (\sqrt{D})i = -\lambda_2$ $\boxed{\text{center}}$



For Hamiltonian system, we $\underline{\text{can not have a sink or a source.}}$

Ex: $\begin{cases} \dfrac{dx}{dt} = -x\cdot\sin y + 2y = f \Leftrightarrow \dfrac{\partial H}{\partial y} \\ \dfrac{dy}{dt} = -\underline{\cos y} = g \Leftrightarrow \underset{= g}{\ominus\dfrac{\partial H}{\partial x}} \end{cases}$ $\Rightarrow$ $\dfrac{\partial f}{\partial x} = \dfrac{\partial^2 H}{\partial x \partial y}$, $\quad \dfrac{\partial g}{\partial y} = -\dfrac{\partial^2 H}{\partial y \partial x}$

$\frac{\partial f}{\partial x} = -\sin y + 0 = -\sin y$

$\frac{\partial g}{\partial y} = \sin y \quad \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 \quad \Rightarrow$

Hamiltonian $\Longleftrightarrow \frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$

$\Updownarrow$

$\boxed{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0}$

$$\begin{cases} \frac{\partial H}{\partial y} = f \\ \frac{\partial H}{\partial x} = -g \end{cases} \qquad \begin{cases} \frac{\partial H}{\partial y} = -x\cdot \sin y + 2y \\ \underline{\frac{\partial H}{\partial x} = +\cos y} \end{cases} \Rightarrow \int \frac{\partial H}{\partial y}\, dy = \int (-x\sin y + 2y)\, dy$$

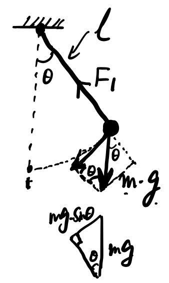
$$\boxed{H(x,y) = \underline{x\cdot\cos y + y^2} + C(x)}$$

$$\frac{\partial H}{\partial x} = \cos y + 0 + C'(x) = -g = \cos y \Rightarrow C'(x) = 0 \Rightarrow C = \text{constant}$$

$$\Rightarrow C = 0.$$

$$\Rightarrow H(x,y) = x\cdot\cos y + y^2$$

Ex: Nonlinear pendulum



Friction $\sim b\cdot$ velocity

$$m\cdot a = F$$

$$m\cdot\frac{d^2}{dt^2}(l\cdot\theta) = \underline{-m\cdot g\cdot\sin\theta - b\cdot\frac{d}{dt}(l\cdot\theta)}.$$

$$\underline{m}\, l\cdot\frac{d^2\theta}{dt^2} + b\cdot l\,\frac{d}{dt}\theta + m\cdot g\cdot\sin\theta = 0.$$

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d}{dt}\theta + \frac{g}{l}\sin\theta = 0 \quad (*)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d}{dt}\theta + \frac{g}{l}\theta = 0}$$

small angle approximation of $(*)$

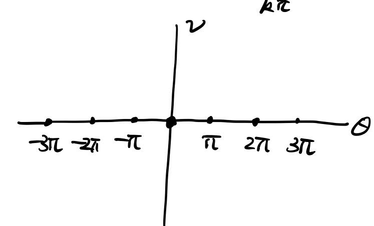
$$v = \frac{d\theta}{dt} \Rightarrow \begin{cases} \frac{d\theta}{dt} = v \\ \frac{dv}{dt} = \frac{d^2\theta}{dt^2} = -\frac{b}{m}\cdot\frac{d\theta}{dt} - \frac{g}{l}\sin\theta = -\frac{b}{m}v - \frac{g}{l}\sin\theta \end{cases}$$

$$\begin{cases} \frac{d\theta}{dt} = v \\ \frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{l}\sin\theta. \end{cases}$$

$x = \theta$
$y = v$

Equilibrium points: $\begin{cases} v = 0 \\ -\frac{bv}{m} - \frac{g}{l}\sin\theta = 0 \end{cases} \Rightarrow \begin{cases} v = 0 \\ \theta = 0, \pm\pi, \pm 2\pi, \cdots \end{cases}$ $k\pi$

$$J = \begin{pmatrix} \frac{\partial f}{\partial\theta} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial\theta} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta & -\frac{b}{m} \end{pmatrix}$$

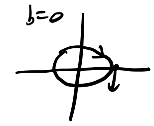


$k = 0, \pm 1, \pm 2, \cdots$

$$J(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos(k\pi) & -\frac{b}{m} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}(-1)^k & -\frac{b}{m} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{m} \end{pmatrix} & k \text{ is even} \\ \begin{pmatrix} 0 & 1 \\ +\frac{g}{l} & -\frac{b}{m} \end{pmatrix} & k \text{ is odd} \end{cases}$$

$b = 0$



• k is even: $T=-\frac{b}{m}$, $D=\frac{g}{l}$ $\qquad \lambda^2 - T\cdot\lambda + D = 0$

$$\Rightarrow \lambda = \frac{T \pm \sqrt{T^2-4D}}{2} = \frac{1}{2}\left(-\frac{b}{m} \pm \sqrt{\left(\frac{b}{m}\right)^2 - \frac{4g}{l}}\right)$$

$$\boxed{b=0:\ \lambda^2+\frac{g}{l}=0 \Rightarrow \lambda = \pm\sqrt{\frac{g}{l}}\,i \text{ (center)}} \qquad \left(\frac{b}{m}\right)^2-\frac{4g}{l}$$

• k is odd: $T=-\frac{b}{m}$, $D=-\frac{g}{l}<0 \Rightarrow$ saddle points.

$$\begin{cases} \frac{d\theta}{dt} = v = f \overset{?}{=} \frac{\partial H}{\partial v} \\ \frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{l}\sin\theta = g \overset{?}{=} -\frac{\partial H}{\partial \theta} \end{cases} \Rightarrow \underbrace{\frac{\partial f}{\partial \theta}}_{=0} + \underbrace{\frac{\partial g}{\partial v}}_{=-\frac{b}{m}-0} \overset{?}{=} 0$$

is Hamiltonian if and only if $b=0$ (no friction)

$$-\frac{b}{m}=0$$

if and only if $b=0$.

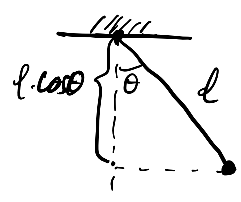
Assume $b=0$.
$$\begin{cases} \frac{\partial H}{\partial v} = v \Rightarrow H(\theta,v) = \frac{1}{2}v^2 + C(\theta) \\ \frac{\partial H}{\partial \theta} = +\frac{g}{l}\sin\theta = C' \Rightarrow C = -\frac{g}{l}\cdot\cos\theta \end{cases}$$

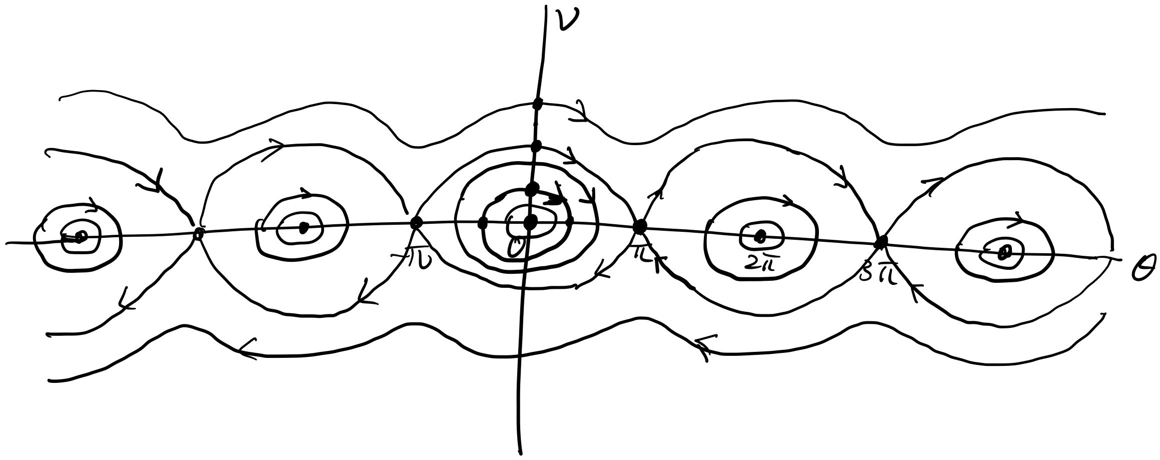
$$m\cdot l^2\cdot H \Leftarrow \boxed{H(\theta,v) = \frac{1}{2}v^2 - \frac{g}{l}\cos\theta}$$

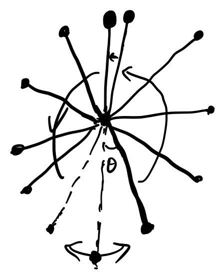
$$m\cdot l^2\cdot H = \frac{1}{2}m\,l^2\dot\theta^2 - (mg)(l\cdot\cos\theta).$$ is a conserved quantity $\boxed{\text{if } b=0}$ = $\boxed{\text{energy}}$

$\frac{1}{2}$ mass × (velocity)^2 + gravity force × height

velocity $= \frac{d}{dt}(l\cdot\theta)$; $\frac{1}{2}$ mass × (velocity)^2 = kinetic energy; gravity force × height = potential energy.







b≠0