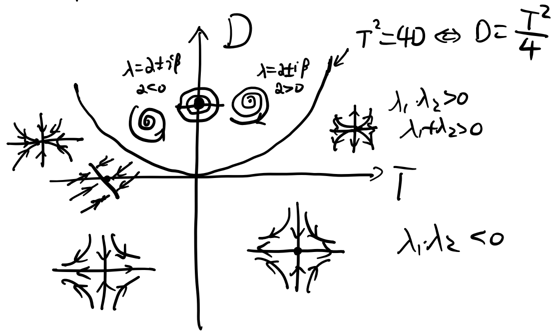
$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

(x_0, y_0) is an equilibrium point: $f(x_0,y_0) = g(x_0,y_0) = 0$.

linearization at (x_0,y_0): $\underline{\dfrac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = J \cdot \begin{pmatrix} u \\ v \end{pmatrix}}$ where $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_0,y_0)}$



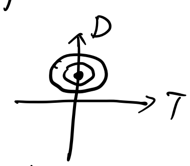
stable under perturbation: saddle, sink, source

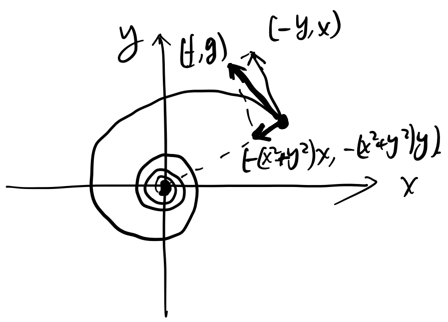
unstable - - - - : center, zero eigenvalue

Ex: $\begin{cases} \dfrac{dx}{dt} = -y - (x^2+y^2)\cdot x = f(x,y) \\ \dfrac{dy}{dt} = x - (x^2+y^2)\cdot y = g(x,y) \end{cases}$

- $(0,0)$ is an equilibrium point
- linearization: $\dfrac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ u \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$T = 0,\ D = 1$





equilibrium for linearization is a center

$$\frac{d}{dt}(x^2+y^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2x\cdot(-y-(x^2+y^2)x) + 2y(x-(x^2+y^2)y)$$
$$= -(x^2+y^2)\cdot 2\cdot(x^2+y^2) = -2\cdot(x^2+y^2)^2$$

$$h(x,y) = x^2+y^2, \quad \frac{d}{dt}h = -2h^2 \Rightarrow \underset{\parallel}{\int -h^{-2}\, dh} = \int 2\,dt$$

$$-\frac{1}{-2+1} h^{-2+1} = 2t + C$$

$$h^{-1} = 2t + C$$

$$\Rightarrow \underset{x^2+y^2}{\underset{\parallel}{h}} = \frac{1}{C+2t} \xrightarrow{t\to+\infty} 0.$$

Hamiltonian system

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$ is a Hamiltonian system if there exists a function $H(x,y)$ such that $f = \frac{\partial H}{\partial y}$, $g = -\frac{\partial H}{\partial x}$

Hamiltonian function

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H}{\partial y} \\ \frac{dy}{dt} = \ominus \frac{\partial H}{\partial x} \end{cases}$$

- Fact: H is a conserved quantity for the system:

$$\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0.$$

Ex: Undamped Oscillator: $my'' + ky = 0 \Leftrightarrow y'' + \frac{k}{m}y = 0.$

$$\Leftrightarrow \frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ y'' = -\frac{k}{m}y \end{pmatrix} = \begin{pmatrix} v \\ -\frac{k}{m}y \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial v} \\ -\frac{\partial H}{\partial y} \end{pmatrix}$$

$$\underline{\int \frac{\partial H}{\partial v} = \int v} \quad \text{and} \quad \underline{\int \frac{\partial H}{\partial y} = +\int \frac{k}{m} y}$$

$$H(y,v) = \frac{1}{2}v^2 + \frac{1}{2}\frac{k}{m}y^2 = \frac{1}{m}\Big(\underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}ky^2}_{\text{elastic energy}}\Big).$$

$H(y(t), v(t)) = \text{const.}$ $\boxed{\text{Conservation of energy}}$

<u>Q1</u>: How to determine whether $\underbrace{\text{an system}}_{\text{autonomous}}$ is Hamiltonian?

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) = \dfrac{\partial H}{\partial y} \\[2ex] \dfrac{dy}{dt} = g(x,y) = -\dfrac{\partial H}{\partial x} \end{cases} \Rightarrow \boxed{\begin{array}{c} \dfrac{\partial f}{\partial x} \\ \| \\ -\dfrac{\partial g}{\partial y} \end{array}} \begin{array}{c} = \dfrac{\partial}{\partial x}\dfrac{\partial H}{\partial y} = \dfrac{\partial^2 H}{\partial x \partial y} \\ \| \\ = \dfrac{\partial}{\partial y}\dfrac{\partial H}{\partial x} = \dfrac{\partial^2 H}{\partial y \partial x} \end{array}$$

$$\boxed{\text{Hamiltonian system} \iff \underline{\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}}}$$

$$\Updownarrow$$

$$\underbrace{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}}_{\text{Divergence of the vector field } \langle f, g \rangle} = 0 \quad \text{(divergence free)}$$

<u>Q2</u>: If Hamiltonian, How do we find H (Hamiltonian fct.)?

$$\frac{\partial H}{\partial y} = f \quad \text{and} \quad \frac{\partial H}{\partial x} = -g$$

Ex:
$$\begin{cases} \dfrac{dx}{dt} = f(y) \\ \dfrac{dy}{dt} = g(x) \end{cases} \qquad \frac{\partial f}{\partial x} = 0 = -\frac{\partial g}{\partial y}$$

is Hamiltonian.

$$\frac{\partial H}{\partial y} = f(y) \Rightarrow H(x,y) = \int f(y)\,dy = F(y) + C(x)$$

$$\frac{\partial H}{\partial x} = -g(x) \qquad \frac{\partial H}{\partial x} = \underbrace{\frac{\partial F}{\partial x}}_{0} + \frac{\partial C}{\partial x} = \frac{\partial C}{\partial x} = -g(x)$$

$$\Downarrow$$

$$C(x) = -\int g(x)\,dx = -G(x).$$

$$\Rightarrow H = F(y) - G(x).$$

$$H(x,y) = \int f(y)\,dy - \int g(x)\,dx.$$

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases}$$ harmonic oscillator undamped.

Ex:
$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + x^2 \end{cases} \longleftrightarrow \underbrace{\frac{d^2x}{dt^2}}_{ma} = -x + \boxed{x^2}$$

nonlinear spring-mass system

$$\frac{\partial H}{\partial y} = y \Rightarrow H = \frac{1}{2}y^2 + C(x)$$

$$\frac{\partial H}{\partial x} = x - x^2 \qquad \frac{\partial H}{\partial x} = \frac{\partial C}{\partial x} = x - x^2 \Rightarrow C(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

$$\Rightarrow \boxed{H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3}$$

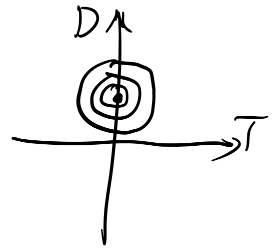
Equilibrium solution $\begin{cases} y = 0 \\ -x + x^2 = 0 \Rightarrow x = 0 \text{ or } 1. \end{cases}$

$$-x + x^2 = x(-1+x)$$

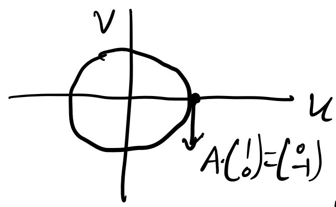
$(0,0)$ and $(1,0)$.

- at $(0,0)$. $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1+2x & 0 \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A$ $\quad T=0,\ D=1$

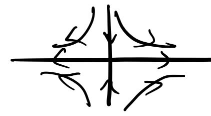


$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$



because of conservation of H along any solution curve. we know that $(0,0)$ is also a center for the original nonlinear system.

- at $(1,0)$ $\quad J(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B$ $\quad T=0,\ D=-1<0$

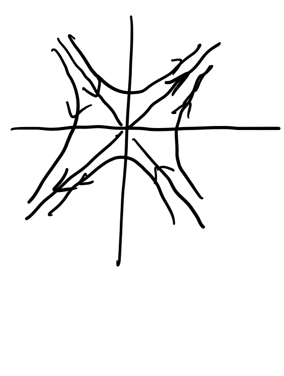


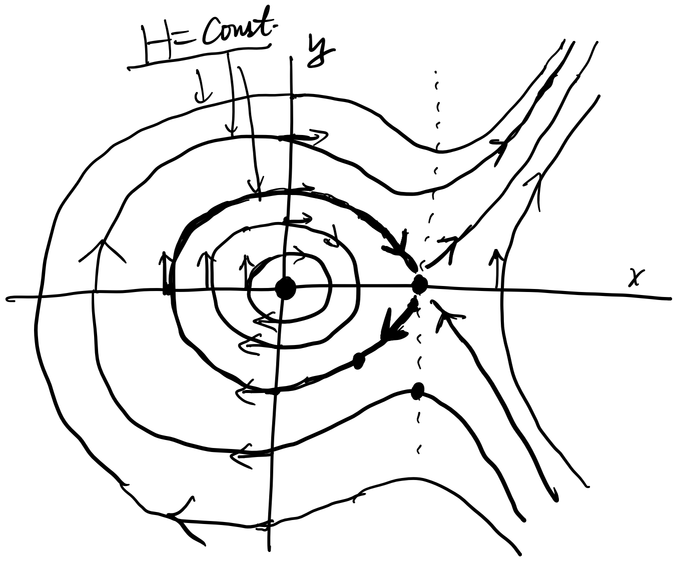
saddle point

$$|B - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = 1 \text{ or } -1$$

$\lambda = 1 > 0$ $\quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -1 < 0$ $\quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$





x- nullcline : $f(x,y) = y = 0$

y- nullcline : $g(x,y) = -x + x^2 = 0 \quad x = 0$ or $x = 1$.