5.1 Equilibrium Point Analysis

Ex:
$$\begin{cases} \dfrac{dx}{dt} = 2\left(1-\dfrac{x}{3}\right)x - x\cdot y \\[2ex] \dfrac{dy}{dt} = \boxed{-2y} + 4xy \end{cases}$$

$\Rightarrow$ x is the prey

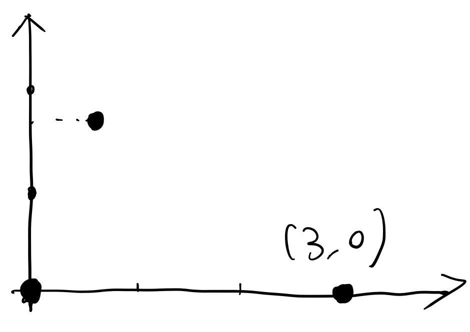
$\Rightarrow$ y is the predator

Equilibrium point:
$$\begin{cases} 2\left(1-\frac{x}{3}\right)x - xy = 0 = x\left(2 - \frac{2x}{3} - y\right) \Rightarrow x=0 \text{ or } \frac{2x}{3}+y=2 \\[1ex] -2y+4xy = 0 = y(-2+4x) \Rightarrow \underline{y=0 \text{ or } x=\frac{1}{2}} \end{cases}$$

$\Rightarrow$ case 1: $x=0,\ y=0$

case 2: $x\neq 0$, $\begin{cases} \frac{2x}{3}+y=2 \\ y=0 \end{cases} \Rightarrow x = 2\times\frac{3}{2} = 3 \Rightarrow (3,0).$

case 3: $x\neq 0$ $\begin{cases} \frac{2x}{3}+y=2 \\ x=\frac{1}{2} \end{cases} \Rightarrow y = 2-\frac{2x}{3} = 2-\frac{1}{3} = \frac{5}{3} \Rightarrow \underline{\left(\frac{1}{2}, \frac{5}{3}\right)}.$

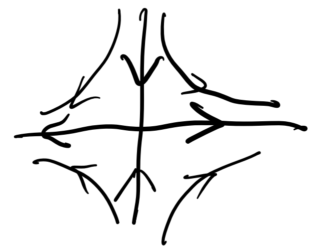


$x \sim 0.01$

$\underline{x^2 = 0.0001 \ll x.}$

$$\begin{cases} \dfrac{dx}{dt} = 2x - \dfrac{2x^2}{3} - xy \\[2ex] \dfrac{dy}{dt} = -2y + 4xy \end{cases} \xrightarrow{\text{approximated by}} \begin{cases} \dfrac{dx}{dt} = 2x \\[2ex] \dfrac{dy}{dt} = -2y \end{cases}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{+2t} \\ c_2 e^{-2t} \end{pmatrix}$$

saddle point

Do change of variable or we can use the linearization near (x_0, y_0).

$$\begin{cases} \frac{dx}{dt} = f(x,y) = \underline{a\cdot(x-x_0) + b\cdot(y-y_0) + O(|x-x_0|^2 + |y-y_0|^2)} \\ \frac{dy}{dt} = g(x,y) = c\cdot(x-x_0) + d(y-y_0) + O(|x-x_0|^2 + |y-y_0|^2) \end{cases}$$

$u = x - x_0$
$v = y - y_0$

$$\begin{cases} \frac{du}{dt} = au + bv + O(|u|^2 + |v|^2) \\ \frac{dv}{dt} = cu + dv + O(|u|^2 + |v|^2). \end{cases}$$

$u, v \sim 0$

$$\rightsquigarrow \begin{cases} \frac{du}{dt} = au + bv \\ \frac{dv}{dt} = cu + dv \end{cases} \qquad \boxed{\frac{d}{dt}U = J\cdot U}$$

linearization at (x_0, y_0).

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{(x,y)=(x_0,y_0)}.$$

Jacobian matrix

$$\begin{cases} \frac{dx}{dt} = 2x - \frac{2x^2}{3} - xy = f(x,y) \\ \frac{dy}{dt} = -2y + 4xy = g(x,y). \end{cases}$$

$$\begin{array}{lcc} & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ & \| & \\ J = & 2-\frac{4x}{3}-y & -x \\ & 4y & -2+4x \\ & \| & \\ & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array}$$

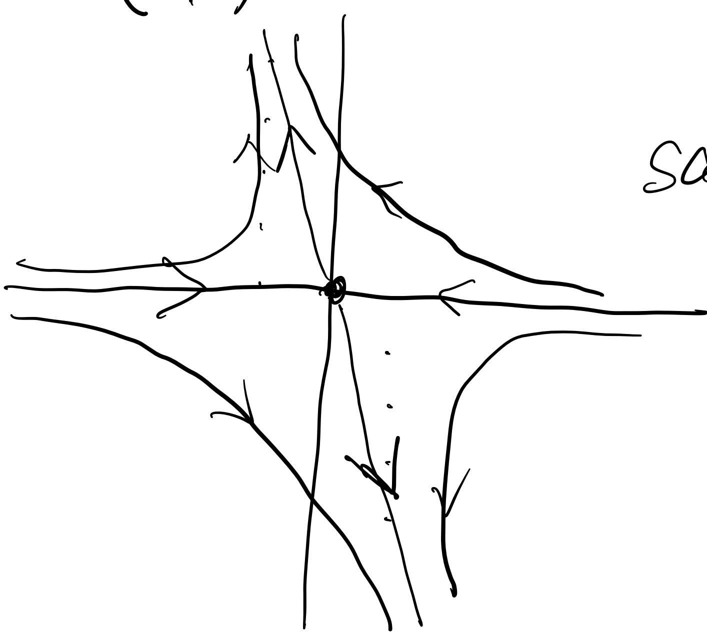
$$J(3,0) = \begin{pmatrix} 2-4-0 & -3 \\ 4\times 0 & -2+4\times 3 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 0 & 10 \end{pmatrix}$$

$\Rightarrow \lambda_1 = -2,\ \underline{\lambda_2 = 10} \Rightarrow$ <u>saddle point</u>

$\Rightarrow \cdot\ (J-\lambda_1 I) = \begin{pmatrix} 0 & -3 \\ 0 & 12 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\cdot\ (J-\lambda_2 10) = \begin{pmatrix} -2-10 & -3 \\ 0 & 10-10 \end{pmatrix} = \begin{pmatrix} -12 & -3 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow v_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$



saddle point

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equilibrium point $\left(\frac{1}{2}, \frac{5}{3}\right)$

$\left(2 - \frac{4x}{2} - y \quad -x \right)$

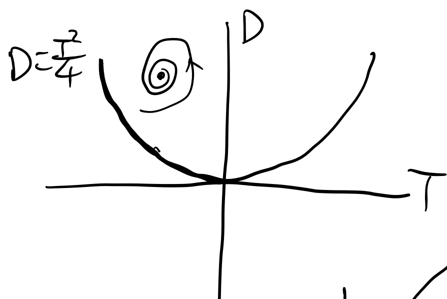
$$J\left(\frac{1}{2},\frac{5}{3}\right)=\begin{pmatrix} \cdots & \cdots \\ 4y & -2+4x \end{pmatrix}\Bigg|_{\left(\frac{1}{2},\frac{5}{3}\right)}$$

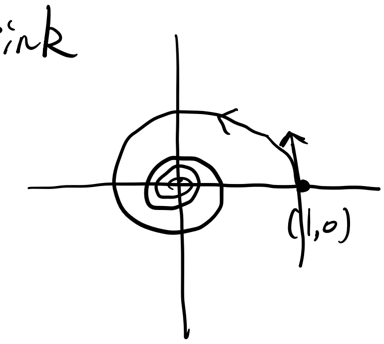
$$=\begin{pmatrix} 2-\frac{2}{3}-\frac{5}{3} & -\frac{1}{2} \\ 4\times\frac{5}{3} & -2+2 \end{pmatrix}=\begin{pmatrix} -\frac{1}{3} & -\frac{1}{2} \\ \frac{20}{3} & 0 \end{pmatrix}$$

$2-\frac{7}{3}=\frac{6-7}{3}$ $\quad T=-\frac{1}{3},\ D=\frac{10}{3}=-\frac{1}{3}\times 0-\left(-\frac{1}{2}\times\frac{20}{3}\right).$

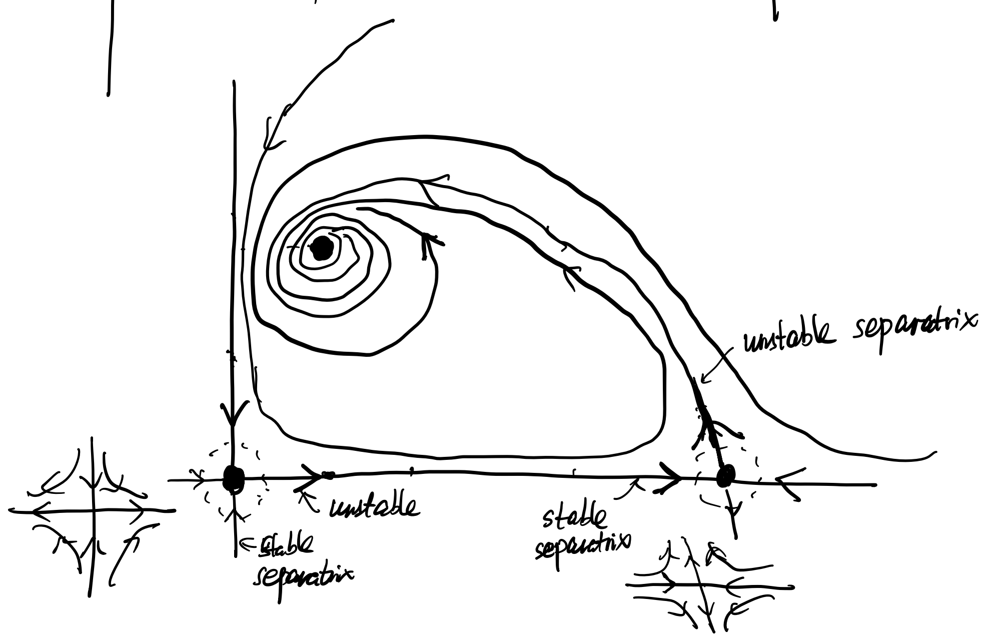
$\lambda=\frac{T\pm\sqrt{T^2-4D}}{2}$ $\quad T^2-4D=\frac{1}{9}-\frac{40}{3}=\frac{1}{9}-\frac{120}{9}=-\frac{119}{9}<0.$

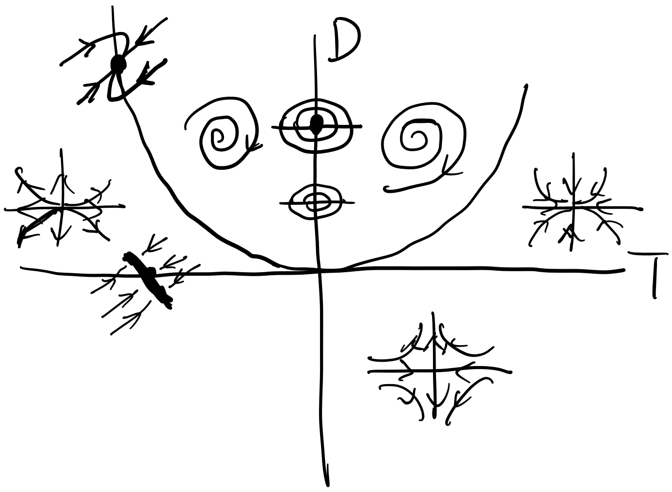
$T=-\frac{1}{3}<0$ $\Rightarrow$ spiraling sink





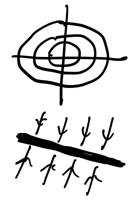
$\begin{pmatrix} -\frac{1}{3} \\ \frac{20}{3} \end{pmatrix}$





unstable type:

- Center
- zero eigenvalue



Nullcline : x-nullcline = the set of points (x,y) where $f(x,y)=0$ $\uparrow\downarrow$

y-nullcline = ... $g(x,y)=0$ $\leftrightarrows$

$$\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}f(x,y)\\g(x,y)\end{pmatrix}$$

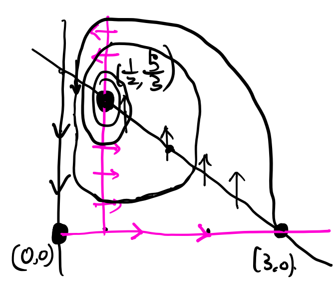
x-nullcline $\cap$ y-nullcline = equilibrium points.

$$\begin{cases}\dfrac{dx}{dt}=2x-\dfrac{2x^2}{3}-xy=f(x,y)\\[2ex]\dfrac{dy}{dt}=-2y+4xy=g(x,y).\end{cases}$$

x-nullcline: $0=f(x,y)=2x-\frac{2x^2}{3}-xy=x\cdot\left(2-\frac{2}{3}x-y\right)$

$\Rightarrow$ $x=0$ or $\frac{2}{3}x+y=2$

$y=2y\cdot(-1+2x)$.



$\frac{2}{3}x+y=2$

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y-nullcline: $0=-2y+4xy=2y\cdot(-1+2x)=0$

$\Rightarrow$ $y=0$ or $x=\frac{1}{2}$