Ex: $\dfrac{dY}{dt} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} Y$

- $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 12 - 12 = \lambda^2 - 8\lambda = \lambda(\lambda-8) = 0$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 = 8$

- $\lambda = 0$ $\quad \underset{A-0\cdot I}{\underset{\parallel}{\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}}} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad x+2y=0 \quad \to \quad v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow Y_1(t) = e^{\lambda_1 t} v_1$

$= e^{0\cdot t} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$

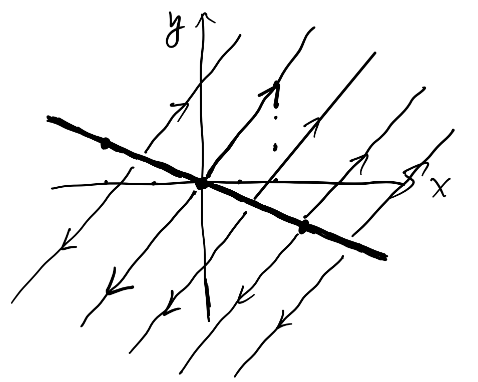
$\dfrac{dY_1}{dt} = 0$

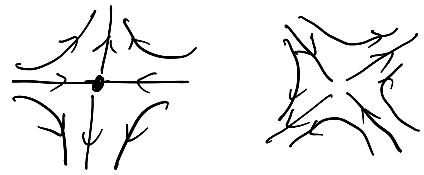
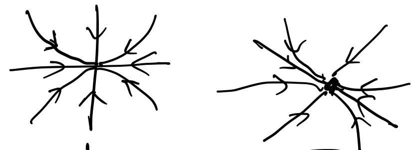
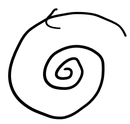
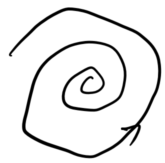
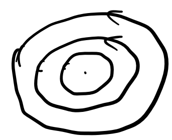
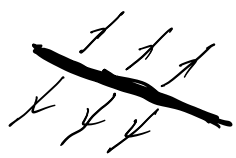
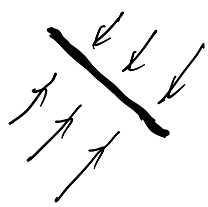
$\lambda = 8 \quad \begin{pmatrix} 2-8 & 4 \\ 3 & 6-8 \end{pmatrix} = \begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix} \to \begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix} \quad 3x - 2y = 0 \Rightarrow v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\Rightarrow Y_2(t) = e^{8t} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

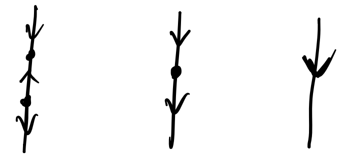
- $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Y(t) = C_1 \cdot Y_1(t) + C_2 \cdot Y_2(t) = \underline{C_1 \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix}} + C_2 \cdot e^{8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

<u>phase plane</u>:



Type	Eigen values	Phase plane
saddle	$\lambda_1 < 0 < \lambda_2$	
sink	$\lambda_1 \leq \lambda_2 < 0$	
source	$0 < \lambda_1 \leq \lambda_2$	
Spiral Sink	$\lambda = a \pm ib$, $b \neq 0$, $a < 0$	
Source	$\lambda = a \pm ib$, $a > 0$, $b \neq 0$	
Center	$\lambda = \pm ib$, $b \neq 0$	
	$\lambda_1 = 0$, $\lambda_2 > 0$	
	$\lambda_1 = 0$, $\lambda_2 < 0$	

$$\frac{dy}{dt} = y(1 - \frac{y}{2}) - a$$



a small a bifurcation a big

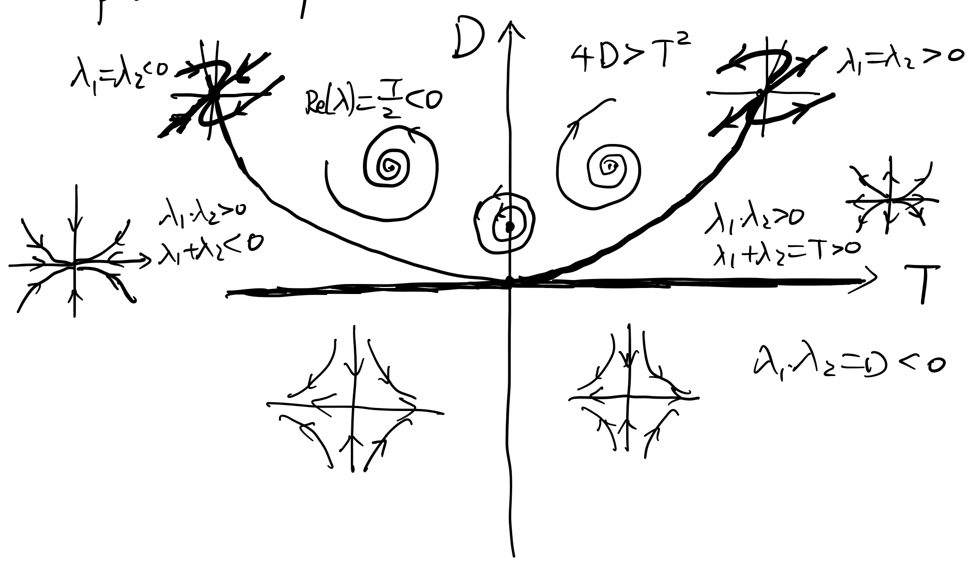
- $\frac{dY}{dt} = A \cdot Y$ $\quad \boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ $\quad T = tr(A) = a+d.$
  $D = det(A) = ad - bc$

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$$

$$= \lambda^2 - T \cdot \lambda + D = (\lambda - \lambda_1) \cdot (\lambda - \lambda_2)$$

$$\Rightarrow \lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \Rightarrow \boxed{\begin{array}{l} \lambda_1 + \lambda_2 = T \\ \lambda_1 \cdot \lambda_2 = D \end{array}}$$

parameter plane = trace-determinant plane.



Ex: $\frac{dY}{dt} = \begin{pmatrix} 2 & 0 \\ a & -3 \end{pmatrix} Y$

$T = 2 - 3 = -1, \quad D = -6 < 0 \Rightarrow$ saddle point.

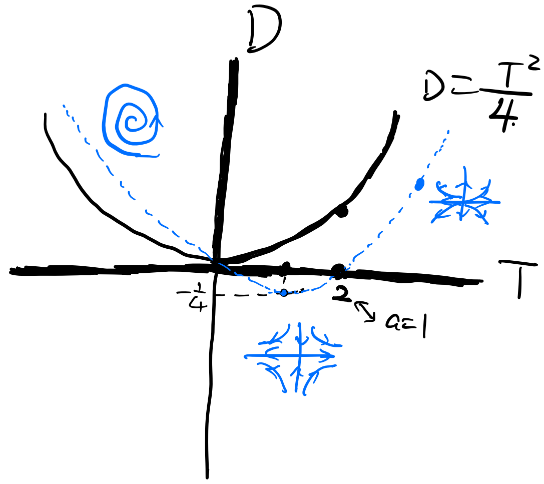
Ex: $\dfrac{dY}{dt} = \begin{pmatrix} a & 1 \\ a & a \end{pmatrix} Y$

$T = 2a$, $\quad D = a^2 - a$

$D = \left(\frac{T}{2}\right)^2 - \frac{T}{2}$

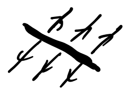
$D = \frac{T^2}{4} - \frac{T}{2} = \frac{1}{4}(T^2 - 2T + 1) - \frac{1}{4}$

$= \frac{1}{4}(T-1)^2 - \frac{1}{4}$



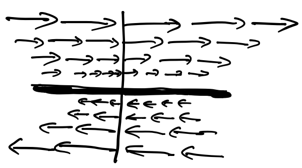
$a > 1$: source

$a = 1$:



$0 < a < 1$: saddle

$a = 0$:



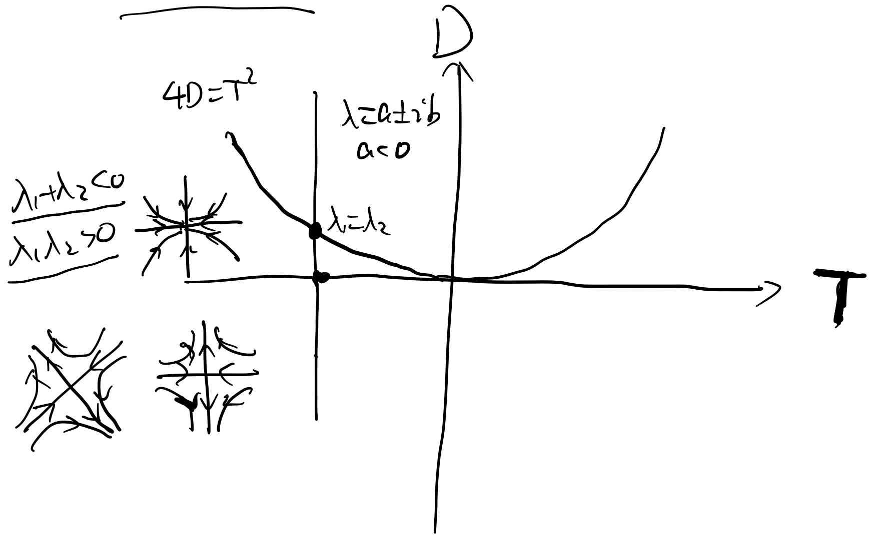
$a < 0$: spiral sink

$\dfrac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y$

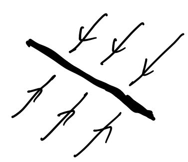
$\begin{cases} \dfrac{dx}{dt} = y \Rightarrow x = c_2 t + c_1 \\ \dfrac{dy}{dt} = 0 \Rightarrow y = c_2 \end{cases}$

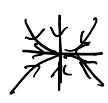
Ex: $\dfrac{dY}{dt} = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix} Y$

$T = -2$, $\quad D = 0-(-2a) = 2a$

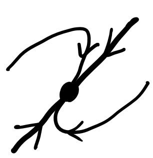


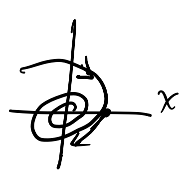
$a < 0$: saddle

$a = 0$: 

$0 < a < a^*$ ($= \frac{1}{2}$): sink  $2a^* = \dfrac{T^2}{4} = \dfrac{(-2)^2}{4} = 1$

$a^* = \frac{1}{2}$

$a = \frac{1}{2}$: 

$a > a^* = \frac{1}{2}$: spiral sink. $\begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ 

Ex: $\frac{dY}{dt} = \begin{pmatrix} -2 & \frac{1}{2} \\ -2 & 0 \end{pmatrix} Y$

$\cdot \begin{vmatrix} -2-\lambda & \frac{1}{2} \\ -2 & 0-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$

$\Rightarrow \lambda = -1.$

$\cdot \begin{pmatrix} -2+1 & \frac{1}{2} \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{1}{2} \\ -2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$ $2x - y = 0$

$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \boxed{Y_1(t) = e^{-t} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$\begin{pmatrix} 2 \\ 4 \end{pmatrix} \neq u_0 \longrightarrow u_1 \to 0$

$\cancel{\lambda}$

$\mathbb{R} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow u_1 = (A - \lambda I) u_0 = \begin{pmatrix} -1 & \frac{1}{2} \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$\Rightarrow Y_2(t) = e^{\lambda t} \cdot (u_1 t + u_0) = e^{-t} \left(\begin{pmatrix} -1 \\ -2 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$

$= e^{-t} \cdot \begin{pmatrix} -t+1 \\ -2t \end{pmatrix}.$

$\Rightarrow \boxed{Y(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \cdot e^{-t} \cdot \begin{pmatrix} -t+1 \\ -2t \end{pmatrix}}.$

$$Y(t) = c_1 \cdot e^{\lambda t} \cdot v_1 + c_2 \cdot e^{\lambda t} (u_1 t + u_0).$$

$v_1 \to 0$ $\qquad$ $u_0 \to u_1 \to 0$

$$= e^{\lambda t} \cdot \Big(\underbrace{(c_1 \cdot v_1 + c_2 u_0)}_{\substack{\| \\ w_0}} + \underbrace{(c_2 u_1) \cdot t}_{\substack{\| \\ w_1}} \Big).$$

$$Y(t) = e^{\lambda t} \cdot (w_0 + w_1 t)$$

$$w_0 \longrightarrow w_1 \longrightarrow 0$$

$$\boxed{\begin{array}{l} \text{If } \lambda \text{ is a repeated eigenvalue and } Y(0) = w_0 \\ Y(t) = e^{\lambda t} \cdot (w_0 + (A - \lambda I) w_0 \cdot t). \end{array}}$$