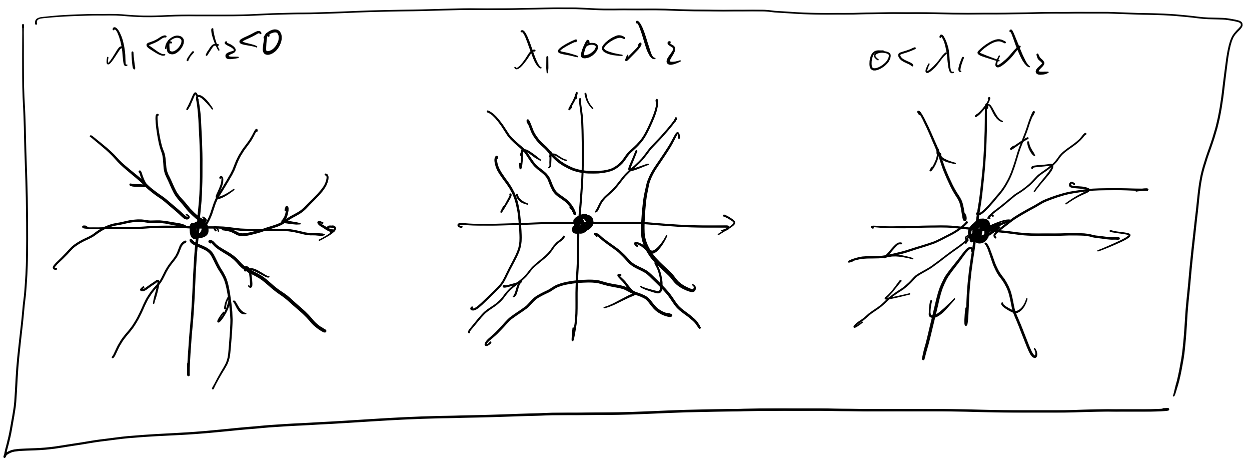
$\frac{dY}{dt} = A \cdot Y$ $\lambda_1, \lambda_2$ are eigenvalues of $A$
(roots to $|A - \lambda I| = 0$)

<u>Case 1</u>: $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ $\Rightarrow$ $Y_1(t) = e^{\lambda_1 t} v_1$

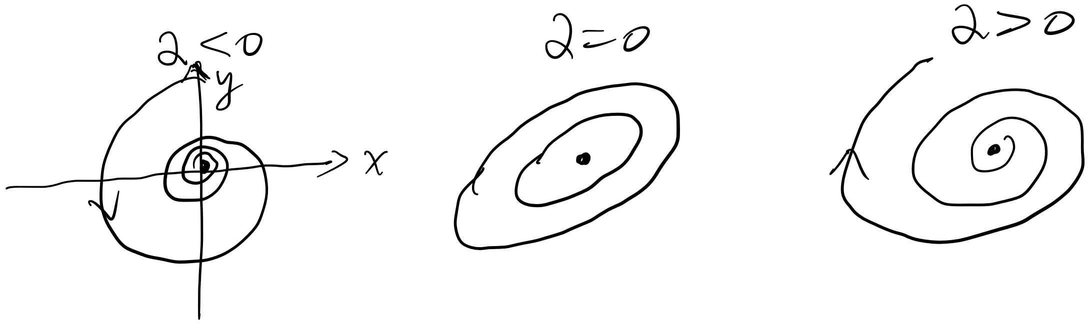
$\lambda_1 \leadsto v_1$, $\lambda_2 \leadsto v_2$ $Y_2(t) = e^{\lambda_2 t} v_2$




<u>Case 2</u>: $\lambda_1 = a + i\beta$, $\beta \neq 0$, $\rightarrow$ $v_1 = \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ c+di \end{pmatrix}$

$\lambda_2 = a - i\beta$

$$\Rightarrow Y_c(t) = e^{\lambda_1 t} v_1 = e^{(a+i\beta)t} \begin{pmatrix} 1 \\ c+di \end{pmatrix} = \underline{Y_1(t)} + i\, \underline{Y_2(t)}$$




**Case 3:** $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$

**case 3.1:** $A$ is a diagonal matrix:

$$\begin{array}{c} A \\ \parallel \\ \lambda \cdot I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \end{array} \Rightarrow A \text{ has 2 linearly independent eigen vectors}$$

$$\begin{cases} \dfrac{dx}{dt} = \lambda x & \Rightarrow x = C_1 \cdot e^{\lambda t} \\[2ex] \dfrac{dy}{dt} = \lambda y & \Rightarrow y = C_2 \cdot e^{\lambda t} \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix} = C_1 e^{\lambda t} \underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + C_2 e^{\lambda t} \underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Case 3.2 : A is not a diagonal matrix

A is not diagonalizable because A has only 1 linearly independent eigenvector.

$\lambda = \lambda_1 = \lambda_2$ is an eigenvalue

the solution space to $(A-\lambda I)\vec{v} = \vec{0}$ is 1-dim

$\parallel$

$\text{Ker}(A-\lambda I)$ = eigenspace associated to $\lambda$

$$\boxed{Y_1(t) = e^{\lambda t}\vec{v}_1} \Leftarrow \vec{v}_1 \to 0 \qquad \vec{u}_0 \not\to \vec{v}_1$$

$$\Leftarrow \vec{u}_0 \to \vec{u}_1 \to 0$$

$$\boxed{Y_2(t) = e^{\lambda t}(\vec{u}_0 + \vec{u}_1 \cdot t)} \text{ is a solution}$$

if $\vec{u}_0 \xrightarrow{A-\lambda I} \underset{\neq 0}{\vec{u}_1} \xrightarrow{A-\lambda I} \vec{0}$

(Choose any $\vec{u}_0 \notin \text{Ker}(A-\lambda I)$, $\vec{u}_1 = (A-\lambda I)\vec{u}_0$ )

$\underline{Ex}$: $\boxed{y'' + 4y' + 4y = 0}$

- $\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda_1 = -2$

  $\quad \overset{\shortparallel}{(\lambda+2)^2} \qquad\qquad \overset{\shortparallel}{\lambda_2}$

$\Rightarrow \boxed{y_1 = e^{-2t}, \quad y_2 = t \cdot e^{-2t}}$

$\Rightarrow y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$

---

- $v = y' \Rightarrow v' = y'' = -4y - 4y'$

  $\qquad\qquad\qquad = -4y - 4v$

$$\begin{cases} y' = v \\ v' = -4y - 4v \end{cases} \Leftrightarrow \boxed{\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}}$$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}. \qquad \frac{d}{dt} Y = A \cdot Y$$

- $|A-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = -2$.

- Find eigenvector:

$2x_1 + x_2 = 0$

$\Updownarrow$

$A - \lambda I = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow Y_1(t) = e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ -2 \cdot e^{-2t} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$

- $u_0 \rightarrow u_1 \rightarrow \vec{0}$. $\quad \in \ker(A-\lambda I)$.

$\ker(A-\lambda I) \ni 2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\ker(A-\lambda I) = \mathbb{R} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ eigenspace

$u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow u_1 = (A-\lambda I)u_0 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$Y_2(t) = e^{\lambda t}(u_0 + t \cdot u_1) = e^{-2t} \cdot \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right) = e^{-2t} \begin{pmatrix} 1+2t \\ -4t \end{pmatrix}$

$$= \begin{pmatrix} e^{-2t} + 2\cdot t\cdot e^{-2t} \\ -4t\cdot e^{-2t} \end{pmatrix} = \begin{pmatrix} \boxed{y_1 + 2\cdot y_2} \\ y_1' + 2y_2' \end{pmatrix}$$

$$\frac{d}{dt}\left[e^{-2t} + 2t\cdot e^{-2t}\right] \qquad\qquad Y_1 + 2\cdot \tilde{Y}_2 = Y_2$$

---

$$\tilde{Y}_2 = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} t\cdot e^{-2t} \\ -2\cdot t e^{-2t} + e^{-2t} \end{pmatrix} = e^{-2t}\cdot\left(\underbrace{\boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}}_{u_0} + t\cdot \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{u_1}\right)$$

$$u_0 + t u_1$$

$$\ker(A-\lambda I) \not\ni \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow (A-\lambda I)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \to 0$$

$$\ker(A-\lambda I) = \mathbb{R}\cdot\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$$Y(t) = C_1\cdot Y_1 + C_2\cdot Y_2 = C_1\cdot Y_1 + C_2(Y_1 + 2\tilde{Y}_2).$$

$$= \tilde{C}_1\cdot Y_1 + \tilde{C}_2\cdot \tilde{Y}_2$$

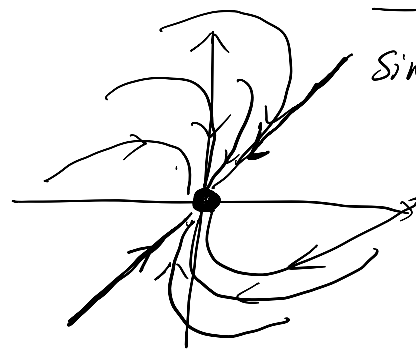
---

$$u_0 = \begin{pmatrix} 10 \\ -1 \end{pmatrix} \notin \mathbb{R}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \mathbb{R}\cdot v_1$$

$$u_0 \longrightarrow u_1 \to 0. \Rightarrow Y_2 = e^{\lambda t}\cdot(u_0 + t u_1).$$

$\lambda_1=\lambda_2=\lambda \Rightarrow \gamma_1(t)=e^{\lambda t}v_1$

$\lambda<0$

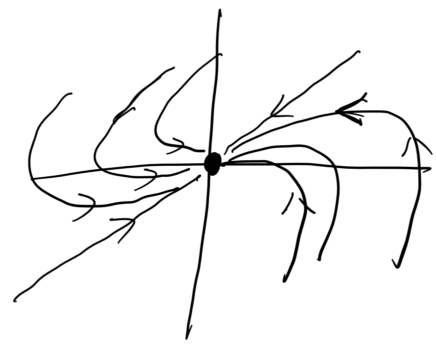


sink

$$\boxed{e^{\lambda t}u_1 t} + e^{\lambda t}u_0 \sim \gamma_2(t)=e^{\lambda t}(u_0+u_1 t)$$

$u_0 \to u_1 \to 0$

$u_1$: eigenvector



$$\boxed{y'' + 4y' + 20y = -3\cdot\sin(2t)}$$ (Review)

- $y'' + 4y' + 20y = 0$

$\lambda^2 + 4\lambda + 20 = 0 \Rightarrow \lambda + 2 = \sqrt{-16} = \pm 4i$

$\lambda^2 + 4\lambda + 20 = (\lambda+2)^2 + 16 \Rightarrow \lambda = -2 \pm 4i$

$\Rightarrow y_1(t) = e^{-2t}\cdot\cos(4t),\ y_2(t) = e^{-2t}\sin(4t)$

$\left(\lambda = \alpha + i\beta \Rightarrow y_1(t) = e^{\alpha t}\cos(\beta t),\ y_2(t) = e^{\alpha t}\sin(\beta t)\right)$

- Complexity: $\boxed{y'' + 4y' + 20y = -3\cdot e^{2it}}$

$e^{2it} = (\cos(2t) + i\sin(2t))$

$$\boxed{y_c(t) = a\cdot e^{2it}}$$

$\Rightarrow y_c' = 2i\cdot a\cdot e^{2it},\ y_c'' = -4a\cdot e^{2it}$

$-4a\cdot e^{2it} + 4\cdot(2i)\cdot a\cdot e^{2it} + 20\cdot a\cdot e^{2it} = -3\cdot e^{2it}$

$(16a + 8i\cdot a)\cancel{e^{2it}} = -3\cancel{e^{2it}}$

$\Rightarrow (16 + 8i)a = -3 \Rightarrow a = -\dfrac{3}{16+8i}$

$$a = -\frac{3}{8}\cdot\frac{1}{2+i} = -\frac{3}{8}\cdot\frac{2-i}{(2+i)\cdot(2-i)} = -\frac{3}{8}\times\frac{2-i}{5}$$

$$(a+i\beta)\cdot(a-i\beta) = a^2+\beta^2 \qquad + \frac{3}{40}(-2+i).$$

$$\underline{y_c(t) = \frac{3}{40}(-2+i)\cdot e^{2it}}$$

$$= \frac{3}{40}(-2+i)\left(\cos(2t) + i\sin(2t)\right)$$

$$= \frac{3}{40}\left(\left(-2\cdot\cos(2t) - \sin(2t)\right) + i\cdot\left(-2\cdot\sin(2t) + \cos(2t)\right)\right)$$

$$\Rightarrow y_p(t) = \mathrm{Im}(y_c(t)) = \frac{3}{40}\left(\cos(2t) - 2\sin(2t)\right).$$

- $y(t) = \underline{y_h} + y_p$

$$= \underbrace{C_1\cdot e^{-2t}\cos(4t) + C_2\cdot e^{-2t}\sin(4t)}_{y_h} + \underbrace{\boxed{\frac{3}{40}(\cos(2t) - 2\sin(2t))}}_{y_p}$$

unforced response

=

underdamped oscillation

$y_h \xrightarrow{t\to+\infty} 0$

$\boxed{\text{steady state}}$

=

forced response

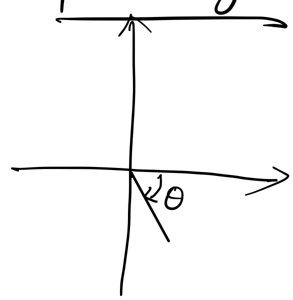
$$y_p = \frac{3}{40}\left(\cos(2t) - 2\sin(2t)\right)$$

$$= \frac{3}{40}\cdot\sqrt{5}\cdot\left(\frac{1}{\sqrt{5}}\cos(2t) - \frac{2}{\sqrt{5}}\sin(2t)\right) = \overbrace{\frac{3\sqrt{5}}{40}}^{\boxed{\text{amplitude}}}\cos(2t - \underline{\theta})$$

$$C_1\cdot\cos(\beta t) + C_2\cdot\sin(\beta t) = \sqrt{C_1^2 + C_2^2}\left(\overbrace{\frac{C_1}{\sqrt{C_1^2+C_2^2}}}^{\cos(\theta)}\cos(\beta t) + \overbrace{\frac{C_2}{\sqrt{C_1^2+C_2^2}}}^{\sin(\theta)}\sin(\beta t)\right)$$

$$= \underbrace{\sqrt{C_1^2 + C_2^2}}_{\text{amplitude}}\cdot\cos(\beta t - \underbrace{\theta}_{\text{phase angle}})$$

$$\cos\theta = \frac{1}{\sqrt{5}}, \quad \sin\theta = -\frac{2}{\sqrt{5}}$$




$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = -2$$

$$\Rightarrow \theta = \tan^{-1}(-2) = -\tan^{-1}(2).$$ phase angle.

$$\theta = 2\pi - \tan^{-1}(2).$$