Ex: $y'' - 2y' + 5y = 0$

Sol: $\lambda^2 - 2\lambda + 5 = 0 = (\lambda-1)^2 + 4 \Rightarrow \lambda - 1 = \sqrt{-4} = \pm 2i$

$\lambda^2 - 2\lambda + 5 = (\lambda^2 - 2\lambda + 1) + 4$

$\Rightarrow \lambda = 1 \pm 2i$

$\Rightarrow y_1(t) = e^t \cos(2t)$ $\quad y_2(t) = e^t \sin(2t)$

$\Rightarrow \boxed{y = c_1 \cdot e^t \cos(2t) + c_2 \cdot e^{2t} \sin(2t) = c_1 y_1 + c_2 y_2}$

$y' = v$, $\quad v' = y'' = -5y + 2y' = -5y + 2v$

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -5y + 2v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -5 & 2 \end{pmatrix}}_{A} \begin{pmatrix} y \\ v \end{pmatrix} . \quad \boxed{\frac{dY}{dt} = A \cdot Y}, \quad Y = \begin{pmatrix} y \\ v \end{pmatrix}$$

- Find eigenvalues/eigenvectors for A

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -5 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = 1 \pm 2i$$

- $\lambda_1 = 1 + 2i$

$$A - \lambda I = \begin{pmatrix} -1-2i & 1 \\ -5 & 2-(1+2i) \end{pmatrix} = \begin{pmatrix} -1-2i & 1 \\ -5 & 1-2i \end{pmatrix} \to \begin{pmatrix} 1+2i & -1 \\ 0 & 0 \end{pmatrix}$$

lead $\quad$ free

$(1+2i)\, y_1 - v_1 = 0$

$$v_1 = \boxed{\begin{pmatrix} 1 \\ 1+2i \end{pmatrix}}, \quad v_2 = \begin{pmatrix} 1-2i \\ 5 \end{pmatrix} = (1-2i) v_1$$

$$\Rightarrow Y_c(t) = e^{\lambda_1 t} v_1 = e^{(1+2i)t} \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$$

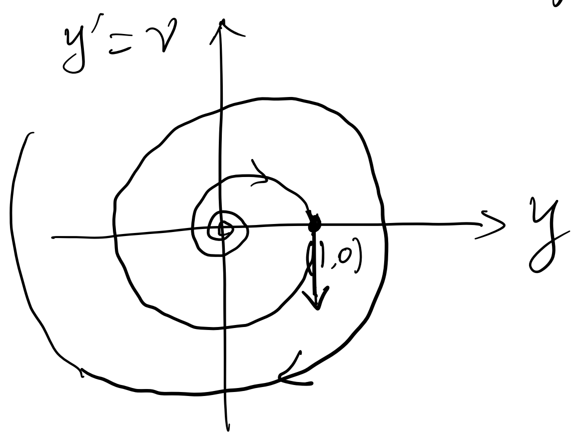
$$= e^t \left(\cos(2t) + i \sin(2t)\right) \cdot \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + i \sin(2t) \\ \cos(2t) - 2\sin(2t) + i \cdot \left(2\cos(2t) + \sin(2t)\right) \end{pmatrix}$$

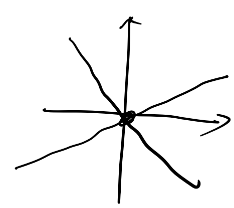
$$= e^t \cdot \begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + i \cdot e^t \begin{pmatrix} \sin(2t) \\ 2\cos(2t) + \sin(2t) \end{pmatrix}$$

$$\Rightarrow Y_1(t) = e^t \begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix}, \quad Y_2(t) = e^t \begin{pmatrix} \sin(2t) \\ 2\cos(2t) + \sin(2t) \end{pmatrix}$$

$$= \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \qquad\qquad \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$$

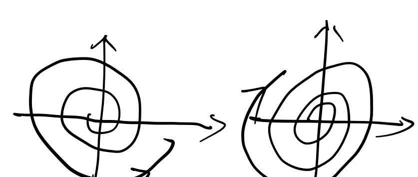
$$\Rightarrow Y(t) = c_1 \cdot Y_1 + c_2 Y_2 = \begin{pmatrix} c_1 y_1 + c_2 \cdot y_2 \\ c_1 y_1' + c_2 y_2' \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$





$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -5 & 2 \end{pmatrix} Y \qquad \begin{pmatrix} 0 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

- $\mathrm{Re}\,\lambda > 0$ spiral source

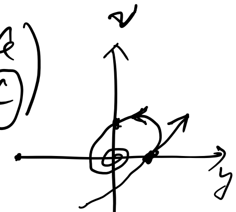


- $\mathrm{Re}\,\lambda < 0$ spiral sink

If $\operatorname{Re}\lambda < 0$

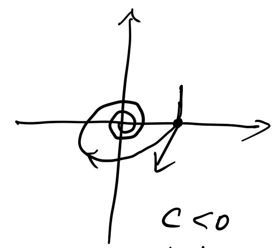
$$A \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}$$

$$\overset{\shortparallel}{\begin{pmatrix}a & b\\ c & d\end{pmatrix}}$$



$c>0$

counterclock



$c<0$

clockwise.

Ex: $\dfrac{dY}{dt} = \begin{pmatrix}-3 & -5\\ 3 & 1\end{pmatrix} Y \qquad Y = \begin{pmatrix}x\\y\end{pmatrix}$

- $\begin{vmatrix}-3-\lambda & -5\\ 3 & 1-\lambda\end{vmatrix} = (\lambda+3)(\lambda-1) - 3\times(-5) = \lambda^2+2\lambda-3+15 = \lambda^2+2\lambda+12 = 0$

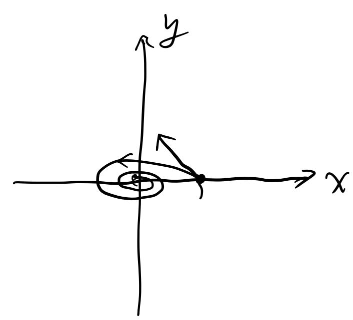
  $\overset{\shortparallel}{|A-\lambda I|}$ $\qquad$ $\overset{\shortparallel}{(\lambda+1)^2+11}$

  $\Rightarrow \lambda = -1 \pm \sqrt{11}\, i$

- $\operatorname{Re}(\lambda) = -1 < 0 \Rightarrow$ spiral sink (as $t \to +\infty$, $Y(t) \to \begin{pmatrix}0\\0\end{pmatrix}$)

- $\begin{pmatrix}-3 & -5\\ 3 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-3\\3\end{pmatrix}$

  counterclockwise.



- calculate eigenvectors:

$\lambda = -1+\sqrt{11}\,i$, $\begin{pmatrix}-3-\lambda & -5\\ 3 & 1-\lambda\end{pmatrix} v = 0 \Rightarrow Y_c(t) = e^{(-1+\sqrt{11}\,i)t} \cdot \begin{pmatrix}1\\z\end{pmatrix}$

$$= e^{-t}\underline{(\cos(\sqrt{11}t) + i\sin(\sqrt{11}t))} \cdot \begin{pmatrix}1\\ \underline{a+ib}\end{pmatrix}$$

$$= e^{-t}\begin{pmatrix}\cos(\sqrt{11}t) + i\sin(\sqrt{11}t)\\ (a\cos(\sqrt{11}t) - b\sin(\sqrt{11}t)) + i\,(b\cdot\cos(\sqrt{11}t) + a\sin(\sqrt{11}t))\end{pmatrix}.$$

$$\Rightarrow \quad Y_1 = \mathrm{Re}(Y_c) = e^{-t}\begin{pmatrix} \cos(\sqrt{11}\,t) \\ a\cos(\sqrt{11}\,t) - b\cdot\sin(\sqrt{11}\,t) \end{pmatrix}$$

$$Y_2 = \mathrm{Im}(Y_c) = e^{-t}\begin{pmatrix} \sin(\sqrt{11}\,t) \\ b\cdot\cos(\sqrt{11}\,t) + a\cdot\sin(\sqrt{11}\,t) \end{pmatrix}.$$

period: $\dfrac{2\pi}{\sqrt{11}}$, frequency $= \dfrac{1}{\text{period}} = \dfrac{\sqrt{11}}{2\pi}$

angular frequency $=$ freq. $\times 2\pi = \sqrt{11}$

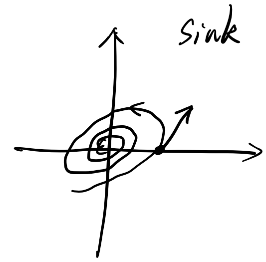
$\lambda = \alpha + i\beta$, $\quad Y_c(t) = e^{\alpha t}\cdot e^{i\beta t}\begin{pmatrix} 1 \\ z \end{pmatrix}$

$$= e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)\begin{pmatrix} 1 \\ \underset{\overset{\shortparallel}{z}}{a+ib} \end{pmatrix}.$$

period: $\dfrac{2\pi}{\beta}$, frequency $= \dfrac{\beta}{2\pi}$

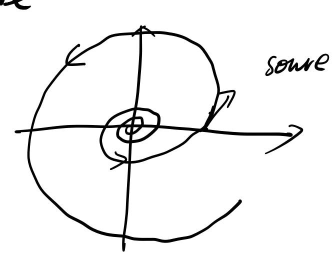
angular freq. $= \beta$.

$$\frac{dY}{dt} = \underset{\overset{\shortparallel}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}}{A}\cdot Y. \qquad \begin{pmatrix} a \\ c \end{pmatrix} = A\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$c>0$ counterclockwise

$c<0$ clockwise



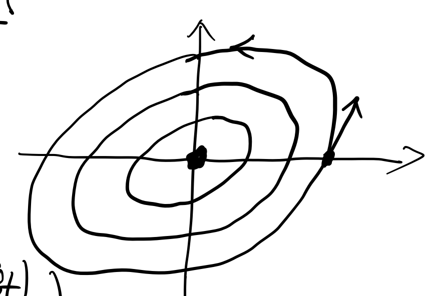
$Re(\lambda)=0 \qquad \lambda = i\cdot\beta \longrightarrow v_c = v_1 + i\cdot v_2$

$$\Rightarrow \quad Y_c(t) = e^{i\beta t} v_c = (\cos(\beta t) + i\sin(\beta t))\cdot(v_1 + i\, v_2)$$

$$= (\cos(\beta t) v_1 - \sin(\beta t) v_2) + i\cdot(\cos(\beta t) v_2 + \sin(\beta t) v_1).$$

$$\Rightarrow \quad Y_1(t) = Re(Y_c(t)), \quad Y_2(t) = Im(Y_c(t))$$

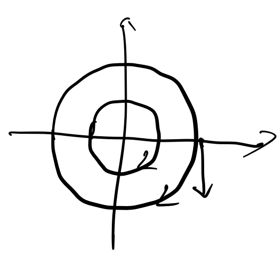
$$\Rightarrow \quad \boxed{Y(t) = C_1\cdot Y_1 + C_2 Y_2}$$



<u>speed</u>: <u>center</u>

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$Y_1(t) = \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}, \quad Y_2(t) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}.$



$x^2 + y^2 = 1.$

$$\frac{dY_1}{dt} = \begin{pmatrix} -\beta\cdot\sin(\beta t) \\ -\beta\cos(\beta t) \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 0 \\ -\beta \end{pmatrix}$$

- $\lambda_1 \neq \lambda_2 \in \mathbb{R} \qquad e^{\lambda_1 t} v_1, \quad e^{\lambda_2 t} v_2$
- $\lambda_1 \neq \lambda_2 \in \mathbb{C} \qquad Re(e^{\lambda_1 t} v_1), \quad Im(e^{\lambda_1 t} v_1).$

- $\lambda_1 = \lambda_2 \in \mathbb{R}$

Ex: $\boxed{\frac{dY}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} Y}$

- $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 + 1 = \underbrace{\lambda^2 - 6\lambda + 9}_{(\lambda-3)^2} = 0$

$\Rightarrow \lambda_1 = 3 = \lambda_2$ (3 is an eig. value of multiplicity 2)

- Find eigenvector:

$A - \lambda I = \begin{pmatrix} 2-3 & 1 \\ -1 & 4-3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Leftrightarrow x_1 - y_1 = 0$

$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow Y_1(t) = e^{\lambda_1 t} v_1 = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$v = \begin{pmatrix} k \\ k \end{pmatrix} = k \cdot v_1$

need find Y_2 that is linearly independent to Y_1.

$Y_2(t) = \boxed{e^{\lambda t} \cdot (t \cdot u_1 + u_0)}$

$e^{\lambda t} t u_1$

$y'' - 6y' + 9y = 0$

$e^{3t}, \boxed{t} e^{3t}$

$$\boxed{Y(t) = e^{\lambda t}(t \cdot u_1 + u_0)}$$

$$Y' = \lambda \cdot e^{\lambda t}(tu_1 + u_0) + e^{\lambda t} \cdot u_1$$

$$= t \lambda \cdot e^{\lambda t} u_1 + e^{\lambda t} \cdot (\lambda u_0 + u_1)$$

$$A \cdot Y = A \cdot e^{\lambda t} \cdot (tu_1 + u_0)$$

$$= t \cdot A \cdot e^{\lambda t} u_1 + A e^{\lambda t} u_0$$

$$\begin{cases} \lambda \cdot \cancel{e^{\lambda t}} u_1 = A \cdot \cancel{e^{\lambda t}} u_1 \\ \cancel{e^{\lambda t}}(\lambda u_0 + u_1) = A \cdot \cancel{e^{\lambda t}} u_0 \end{cases} \Leftrightarrow \begin{cases} A u_1 = \lambda u_1 \\ A u_0 = \lambda u_0 + u_1 \end{cases}$$

$$\Updownarrow$$

$$\boxed{Y(t) = e^{\lambda t}(tu_1 + u_0)} \Leftrightarrow \begin{cases} (A - \lambda I) u_1 = 0 \\ (A - \lambda I) u_0 = u_1 \end{cases}$$

is a solution

$\boxed{\text{Chain}}$ $\quad u_0 \xrightarrow{A-\lambda I} u_1 \xrightarrow{A-\lambda I} 0$

$$\Updownarrow$$

u_1 is an eigenvector.

want: $u_1 \neq 0$

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}. \qquad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{A-3I} 0$$

$$A - \lambda I = \begin{pmatrix} 2-3 & 1 \\ -1 & 4-3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Choose $u_0 \notin \langle v_1 \rangle$ eigenspace

$$u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{A-\lambda I} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \xrightarrow{A-\lambda I} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$u_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = (-1)\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Y_2(t) = \underline{e^{\lambda t}(u_1 t + u_0)} = e^{3t}\left(\begin{pmatrix} -1 \\ -1 \end{pmatrix}\cdot t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$Y_2(t) = e^{3t}\begin{pmatrix} -t+1 \\ -t \end{pmatrix}.$$

$$Y_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\Rightarrow Y(t) = C_1 \cdot Y_1(t) + C_2 \cdot Y_2(t).$$