$\frac{dY}{dt} = A \cdot Y$ $\quad$ A 2x2 matrix

- general solution: $Y(t) = C_1 Y_1 + C_2 Y_2$ where Y_1 and Y_2 are 2 linearly independent solutions.

$\Updownarrow$

$Y_1(0)$ and $Y_2(0)$ are not parallel.

- To find Y_1 and Y_2, calculate eigenvalues and eigenvectors for A

$$\lambda_1 \to v_1 \Rightarrow Y_1(t) = e^{\lambda_1 t} v_1$$
$$\lambda_1 \neq \lambda_2$$
$$\lambda_2 \to v_2 \Rightarrow Y_2(t) = e^{\lambda_2 t} v_2$$

linearly independent.

Ex: $\frac{dY}{dt} = \underbrace{\begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}}_{= A} Y$, $\quad Y(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

- Calculate eigenvalues/eigenvectors:

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 1 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 12 - 2 = \underbrace{\lambda^2 + 7\lambda + 10}_{= (\lambda+2)(\lambda+5)} = 0$$

$$\Rightarrow \lambda_1 = -2 \neq \lambda_2 = -5$$

- $\lambda_1 = -2$: $(A - \lambda I) = \begin{pmatrix} -4-(-2) & 1 \\ 2 & -3-(-2) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$

leading $\quad$ free

$x_1 - \frac{1}{2} y_1 = 0$

$\rightsquigarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector.

$\rightsquigarrow Y_1 = e^{\lambda_1 t} v_1 = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

- $\lambda_2 = -5$ $\quad (A - \lambda I) = \begin{pmatrix} -4-(-5) & 1 \\ 2 & -3-(-5) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$x_1 + y_1 = 0$

$\rightsquigarrow v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightsquigarrow Y_2 = e^{-5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad Y(t) = C_1 \cdot e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} + C_2 \cdot e^{-5t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

$$Y(0) = \begin{pmatrix}2\\1\end{pmatrix} = C_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} + C_2 \cdot \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}\cdot\begin{pmatrix}C_1\\C_2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix}C_1\\C_2\end{pmatrix} = \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}^{-1}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{1-(-2)}\begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}$$

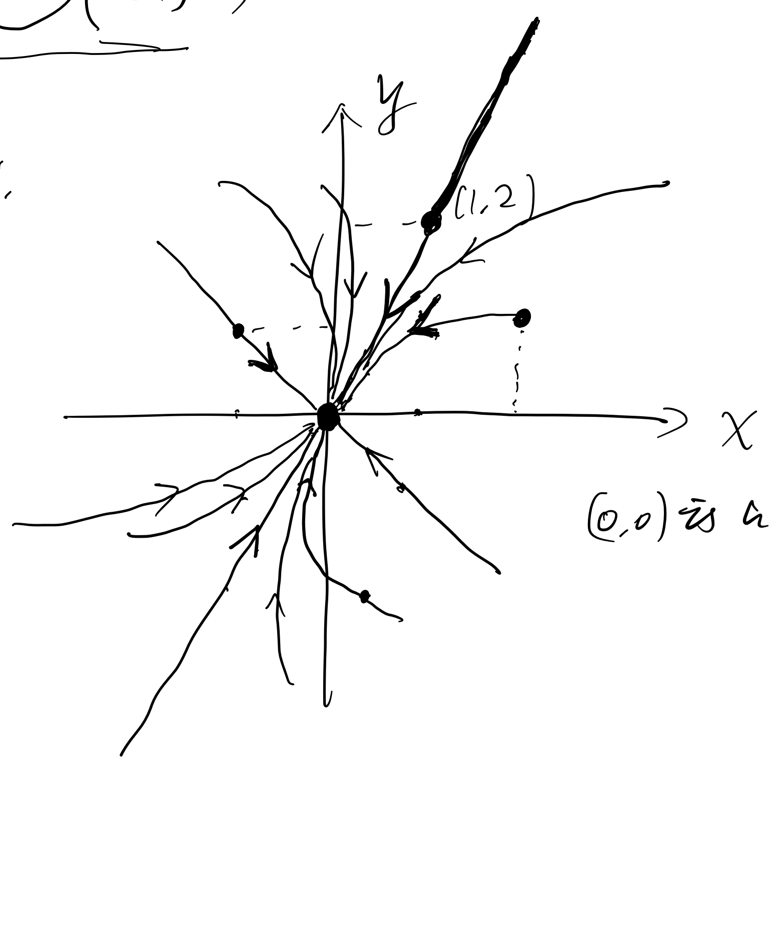
$$\begin{pmatrix}a & b\\c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\-c & a\end{pmatrix} \qquad = \frac{1}{3}\cdot\begin{pmatrix}3\\-3\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

$$\Rightarrow \quad Y(t) = 1 \cdot e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} - e^{-5t}\begin{pmatrix}-1\\1\end{pmatrix} = \boxed{\begin{pmatrix}e^{-2t} + e^{-5t}\\2e^{-2t} - e^{-5t}\end{pmatrix}}.$$

$$Y_1 = C_1 e^{-2t}\begin{pmatrix}1\\2\end{pmatrix}, \quad Y_2 = C_2 e^{-5t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

phase plane:

$$Y = \begin{pmatrix}x\\y\end{pmatrix}$$



(0,0) is a sink

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-2t} + e^{-5t} \\ 2e^{-2t} - e^{-5t} \end{pmatrix} \sim \boxed{\begin{pmatrix} e^{-2t} \\ 2\cdot e^{-2t} \end{pmatrix}} + O(e^{-5t})$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4\cdot e^{-2t} + 5\cdot e^{-5t}}{-2e^{-2t} - 5\cdot e^{-5t}} \xrightarrow{t\to+\infty} 2$$

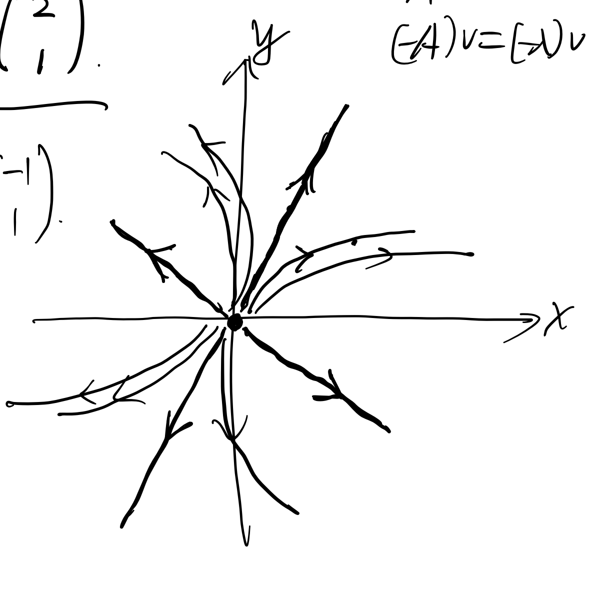
$$= \frac{\cancel{e^{-2t}}\cdot(-4 + 5\cdot e^{-3t \to 0})}{\cancel{e^{-2t}}(-2 - 5\cdot \underbrace{e^{-3t}}_{\to 0})}$$

$Av = \lambda v$

$(-A)v = (-\lambda)v$

$$\frac{dY}{dt} = \begin{pmatrix} +4 & -1 \\ -2 & +3 \end{pmatrix} Y, \quad \underline{Y(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}.$$

$$Y_1(t) = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad Y_2(t) = e^{5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

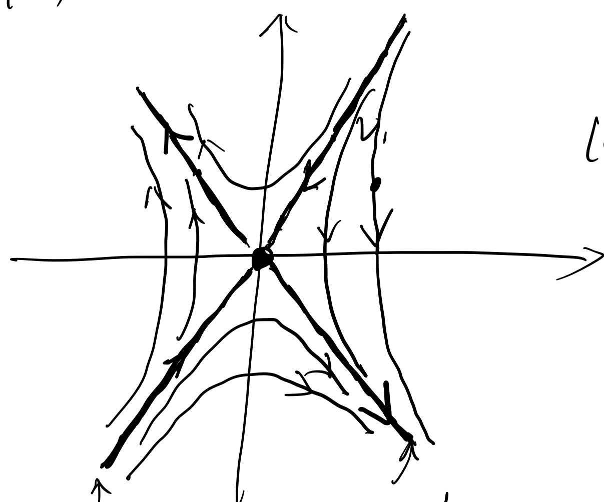


(0,0) is a source.

The other case: $\lambda_1 < 0 < \lambda_2$

$$\begin{matrix} \lambda_1 & < 0 < & \lambda_2 \\ \downarrow & & \downarrow \\ v_1 & & v_2 \end{matrix}$$

$$\gamma_1(t) = e^{\lambda_1 t} v_1 \ , \quad \gamma_2(t) = e^{\lambda_2 t} v_2$$



$(0,0)$ is a saddle point

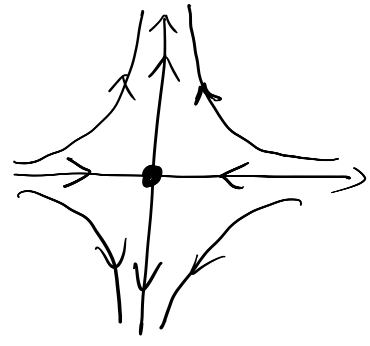
Very special case: $\lambda_1 < 0 < \lambda_2$

$$\begin{matrix} \lambda_1 & < 0 < & \lambda_2 \\ \downarrow & & \downarrow \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix}$$

$$\gamma(t) = c_1 \cdot e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \cdot e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$\frac{d}{dt}\gamma = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \gamma.$$

decoupled case.



Ex: $\frac{dY}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} Y \qquad Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

$$\cdot \; |A - \lambda I| = \underbrace{\begin{vmatrix} 2-\lambda & 2 \\ -4 & 6-\lambda \end{vmatrix}} = \lambda^2 - 8\lambda + 12 + 8 = \lambda^2 - 8\lambda + 20$$

$$|A - \lambda I| = 0$$

$$\lambda^2 - 8\lambda + 20 = (\lambda^2 - 8\lambda + 16) + 4 = (\lambda-4)^2 + 4.$$

$$\Rightarrow (\lambda - 4)^2 = -4 \Rightarrow \lambda - 4 = \pm 2i$$

$$\Rightarrow \lambda = 4 \pm 2i.$$

$$\cdot \; \underline{\lambda_1 = 4 + 2i} \quad \begin{pmatrix} 2-(4+2i) & 2 \\ -4 & 6-(4+2i) \end{pmatrix} = \begin{pmatrix} -2-2i & 2 \\ -4 & 2-2i \end{pmatrix}$$

$$(-2-2i)(2-2i) + 8 = -8 + 8 = 0.$$

$$\rightarrow \begin{pmatrix} 2 & \frac{2-2i}{-2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1+i \\ 0 & 0 \end{pmatrix} \qquad 2x_1 + (-1+i)y_1 = 0.$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{leading} & \text{free} \end{array}$$

$$\rightarrow \boxed{\begin{pmatrix} 1-i \\ 2 \end{pmatrix} = v_1}$$

$$2x_1 + (-2+2i) = 0$$

$$\Rightarrow x_1 = \frac{2-2i}{2} = 1-i$$

$$\Rightarrow Y_1(t) = e^{\lambda_1 t} v_1 = e^{(4+2i)t} \begin{pmatrix} 1-i \\ 2 \end{pmatrix} = \underline{u(t) + i v(t)}.$$

$$\lambda_2 = 4-2i, \quad v_2 = \overline{v_1}, \quad Y_2(t) = \overline{Y_1(t)} = u(t) - i v(t)$$

$$\lambda_2 = \overline{\lambda_1} = \overline{4+2i}$$

$$\begin{pmatrix} Av = \lambda v \\ \Updownarrow \\ A\overline{v} = \overline{\lambda}\,\overline{v} \end{pmatrix} \Longrightarrow \frac{Y_1+Y_2}{2} = u(t), \quad \frac{Y_1-Y_2}{2i} = v(t)$$

are solutions.

linearly independent

$$e^{(4+2i)t}\begin{pmatrix} 1-i \\ 2 \end{pmatrix} = \boxed{e^{4t}[\cos(2t) + i\sin(2t)]}\begin{pmatrix} 1-i \\ 2 \end{pmatrix}$$

$$= e^{4t}\begin{pmatrix} \cos(2t)+\sin(2t) + i(\sin(2t)-\cos(2t)) \\ 2\cos(2t) + i\cdot 2\sin(2t) \end{pmatrix}.$$

$$\left(\begin{aligned} (a+bi)(c+di) &= ac + adi + bci + bd\,(i^2 = -1) \\ &= (ac-bd) + (ad+bc)i \end{aligned} \right)$$

$$\Rightarrow u(t) = e^{4t}\begin{pmatrix} \cos(2t)+\sin(2t) \\ 2\cos(2t) \end{pmatrix}$$

$$v(t) = e^{4t}\begin{pmatrix}\sin(2t)-\cos(2t)\\ 2\cdot\sin(2t)\end{pmatrix}$$

$$\Rightarrow Y(t) = c_1\cdot u(t) + c_2\cdot v(t)$$

$$Y(0) = \begin{pmatrix}1\\1\end{pmatrix} = c_1\cdot u_1(0) + c_2 v(0) = c_1\cdot\begin{pmatrix}1\\2\end{pmatrix} + c_2\cdot\begin{pmatrix}-1\\0\end{pmatrix}$$

$$1 = \begin{pmatrix}c_1-c_2\\ 2c_1\end{pmatrix} = \begin{pmatrix}1 & -1\\ 2 & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

$$\Rightarrow 2c_1 = 1 \Rightarrow c_1 = \frac{1}{2} \Rightarrow c_2 = c_1 - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\Rightarrow Y(t) = \frac{1}{2}\cdot e^{4t}\begin{pmatrix}\cos(2t)+\sin(2t)\\ 2\cos(2t)\end{pmatrix} - \frac{1}{2}e^{4t}\begin{pmatrix}\sin(2t)-\cos(2t)\\ 2\sin(2t)\end{pmatrix}$$

$$= e^{4t}\begin{pmatrix}\cos(2t)\\ \cos(2t)-\sin(2t)\end{pmatrix}$$

$$\lambda = \alpha \pm i\beta, \quad v = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$Y_1(t) = e^{\lambda_1 t} v = \underline{\underline{e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

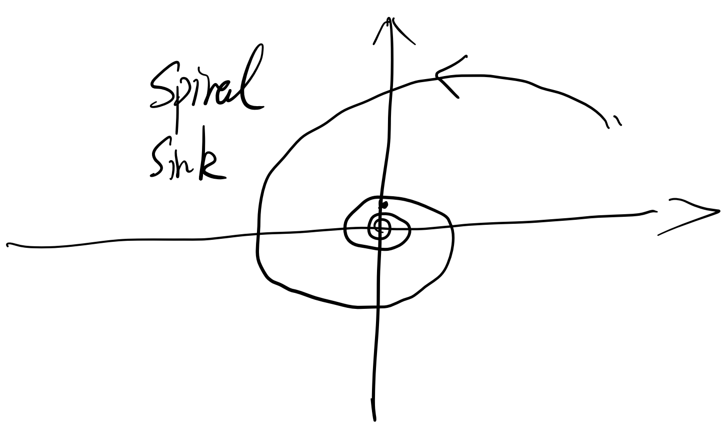
$$\|$$

$$u + iv$$

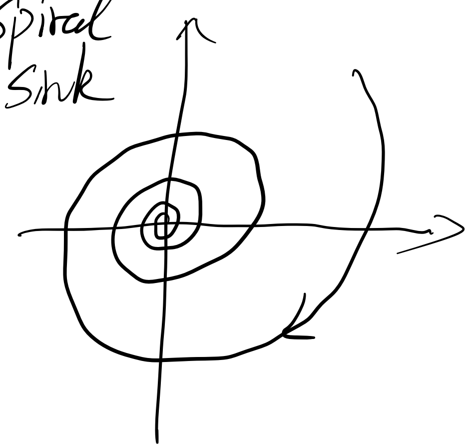
$$\begin{matrix} u = \\ v = \end{matrix} \begin{pmatrix} k_1 e^{\alpha t}\cos(\beta t) + k_2 e^{\alpha t}\sin(\beta t) \\ k_3 e^{\alpha t}\cos(\beta t) + k_4 e^{\alpha t}\sin(\beta t) \end{pmatrix} e^{\alpha t}$$

If $\alpha < 0$, $u, v \to 0$ as $t \to +\infty$

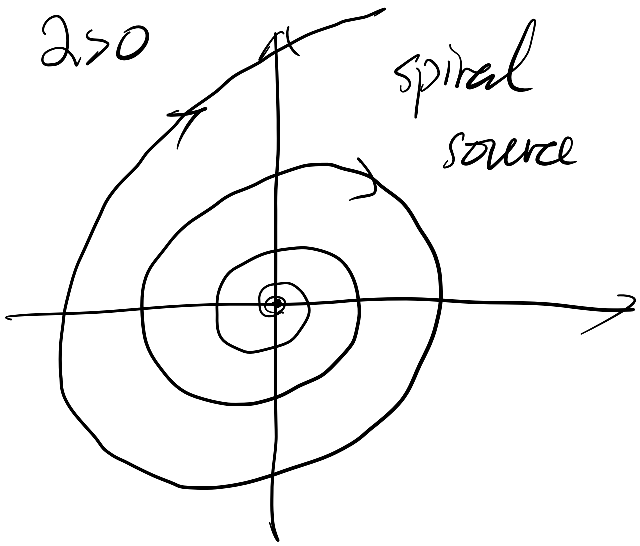
Spiral Sink



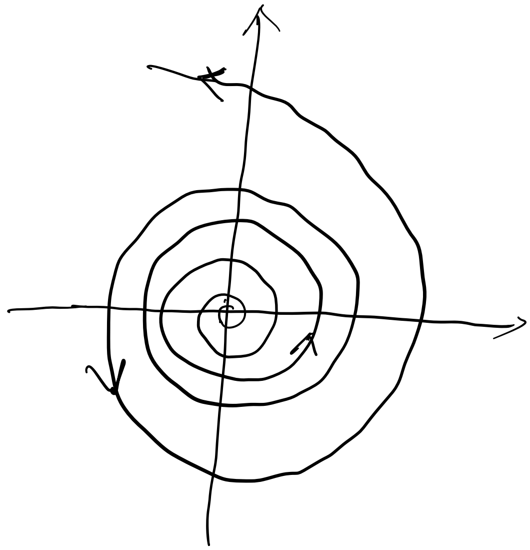
Spiral Sink



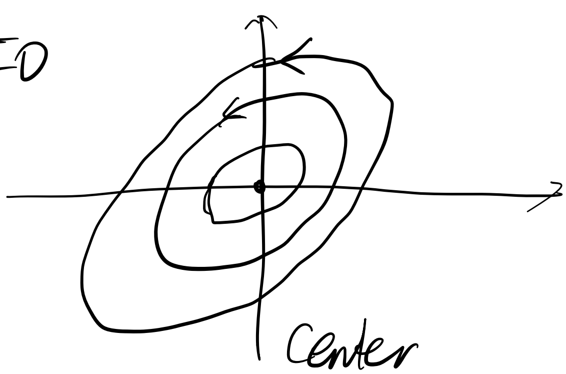
$\lambda > 0$



spiral source



$\lambda = 0$



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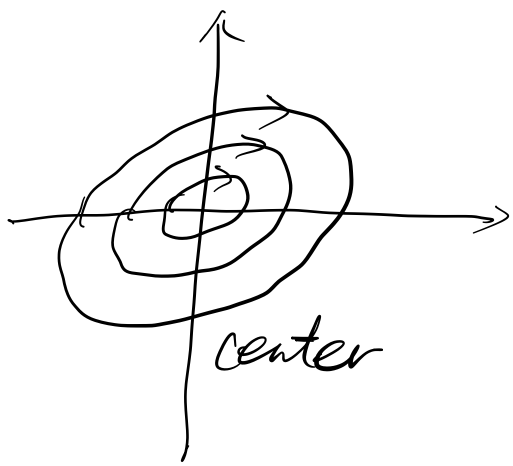

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