$$\frac{dY}{dt} = A \cdot Y \iff \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

Thm: The general solution to $\frac{dY}{dt} = A \cdot Y$.

$$Y(t) = C_1 \cdot Y_1(t) + C_2 \, Y_2(t)$$

where $Y_1 = Y_1(t)$ and $Y_2 = Y_2(t)$ are $\boxed{\text{linearly independent}}$ solutions.

$\Updownarrow$

$Y_1(0)$ and $Y_2(0)$ are linearly independent

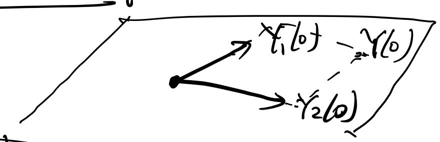
Pf: Assume Y_1, Y_2 are two solutions.

$\boxed{Y(t)}$ is another solution. Want to find C_1 and C_2 s.t. $\boxed{Y = C_1 Y_1 + C_2 Y_2}$

$\Rightarrow Y(0) = C_1 \cdot Y_1(0) + C_2 \, Y_2(0)$ Goal

Because $Y_1(0)$ and $Y_2(0)$ are $\boxed{\text{linearly independent vector}}$ in $\mathbb{R}^2$,

$\{Y_1(0), Y_2(0)\}$ form a basis of $\mathbb{R}^2$.

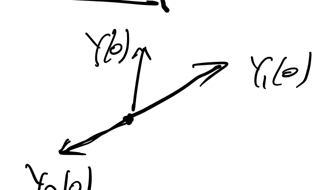


$\Rightarrow \exists$ a unique C_1 and C_2 s.t. $Y(0) = C_1 Y_1(0) + C_2 Y_2(0)$.

$\Rightarrow \tilde{Y}(t) = C_1 Y_1(t) + C_2 Y_2(t)$. a solution by $\boxed{\text{superposition Principle}}$

$Y(0) = C_1 \cdot Y_1(0) + C_2 \cdot Y_2(0)$, $\boxed{\tilde{Y}(0) = C_1 Y(0) + C_2 Y_2(0) = Y(0)}$

Then by $\boxed{\text{Uniqueness Theorem,}}$ $Y = \boxed{\tilde{Y} = C_1 Y_1(t) + C_2 \cdot Y_2(t)}$ ∎



$$\boxed{\frac{dY}{dt} = A\cdot Y}$$

$$\frac{dy}{dt} = a\cdot y \Rightarrow \frac{dy}{y} = a\,dt \Rightarrow \boxed{y = C e^{at}}$$

$$\ln y = at + C_1$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Y(t) = e^{\lambda t}\cdot v \qquad v = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \neq 0$$

$$\frac{dY}{dt} = \boxed{\lambda\cdot e^{\lambda t}\cdot v = A\cdot e^{\lambda t}\cdot v} \overset{e^{\lambda t}\neq 0}{\Longleftrightarrow} \boxed{A v = \lambda\cdot v}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

$\Updownarrow$

$0 \neq v$ is an eigenvector for A with the eigenvalue λ.

Ex:

$$\boxed{\begin{cases} \frac{dx}{dt} = -2x - 2y \\ \frac{dy}{dt} = -2x + y \end{cases}}$$

$\Updownarrow$

$$\frac{dY}{dt} = \underbrace{\begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix}}_{=A}\begin{pmatrix} x \\ y \end{pmatrix}$$

Find eigenvalues and eigenvectors for A

Find λ and $v \neq 0$ s.t. $A v - \lambda\cdot v = 0$ $\qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\boxed{A - \lambda I}\cdot v = 0$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$\boxed{\text{Characteristic}}$ polynomial of A $\qquad (\lambda+2)(\lambda-1) - 4$

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (-2-\lambda)\cdot(1-\lambda) - (-2)\cdot(-2)$$

$$|A-\lambda I| = \lambda^2 + 2\lambda - \lambda - 2 - 4 = \underset{\underset{(\lambda+3)(\lambda-2)}{\parallel}}{\lambda^2+\lambda-6} = 0$$

$$\Rightarrow \boxed{\lambda_1 = -3,\ \lambda_2 = 2}$$

- $\lambda_1 = -3$, $A - \lambda I = \begin{pmatrix} -2-(-3) & -2 \\ -2 & 1-(-3) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

$$\downarrow$$

$$\boxed{v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}} \Leftarrow x_1 = 2, y_1 = 1 \Leftarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{y_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2y_1 \\ y_1 \end{pmatrix}} \qquad \Updownarrow \quad x_1 - 2y_1 = 0$$

$$\Rightarrow \boxed{Y_1(t) = e^{\lambda_1 t} v_1 = e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

- $\lambda_2 = 2$ $A - \lambda I = \begin{pmatrix} -2-2 & -2 \\ -2 & 1-2 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}$

$$\downarrow$$

$$v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Leftarrow \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \Leftrightarrow x_1 + \frac{1}{2}y_1 = 0$$

$$\begin{matrix} \uparrow & \uparrow \\ \text{leading} & \text{free} \end{matrix}$$

$$\Rightarrow \boxed{Y_2(t) = e^{\lambda_2 t} v_2 = e^{2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

- $Y_1(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = v_1$ $Y_2(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = v_2$

$$\lambda_1 = -3 \neq 2 = \lambda_2$$

$\boxed{\text{Fact}}$: Eigenvectors associated to different eigenvalues are linearly independent.

$\Rightarrow$ General solution

$$Y(t) = C_1 Y_1(t) + C_2 Y_2(t)$$

$$= C_1 \cdot e^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \cdot e^{2t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

$$= \begin{pmatrix} 2C_1 e^{-3t} - C_2 \cdot e^{2t} \\ C_1 e^{-3t} + 2C_2 \cdot e^{2t} \end{pmatrix}$$

Add Initial Condition: $Y(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

$$\begin{pmatrix} 2C_1 - C_2 \\ C_1 + 2C_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \frac{1}{2\times 2 - 1\times(-1)}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$= \frac{1}{5}\cdot\begin{pmatrix} 4+5 \\ -2+10 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 9 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$\Rightarrow \quad Y(t) = \frac{9}{5} e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{8}{5} e^{2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$ is the unique

solution to the IVP

$$\begin{cases} \frac{dx}{dt} = -2x - 2y \\ \frac{dy}{dt} = -2x + y \end{cases}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$