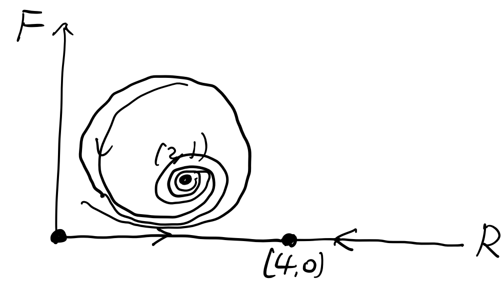
$$\begin{cases} \dfrac{dR}{dt} = \underline{2R\left(1-\dfrac{R}{4}\right)} - RF \\[2ex] \dfrac{dF}{dt} = -F + \dfrac{1}{2}RF \end{cases}$$



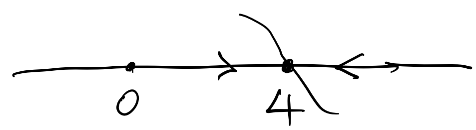
Find equilibrium solution:

$$\begin{cases} 2R\left(1-\frac{R}{4}\right) - RF = 0 = R\cdot\left(2-\frac{R}{2}-F\right) \Rightarrow R=0 \text{ or } \boxed{\frac{R}{2}+F=2}. \\[2ex] -F+\frac{1}{2}RF = 0 = F\left(-1+\frac{R}{2}\right) \Rightarrow F=0 \text{ or } R=2. \end{cases}$$

$$\Rightarrow (F=0,\ R=0) \text{ or } (F=0 \text{ and } R=4) \text{ or } \left(F\neq 0,\ R=2,\ F=2-\frac{R}{2}=1\right).$$

$$(\underset{\underset{R}{\uparrow}}{0}, \underset{\underset{F}{\uparrow}}{0}) \qquad (4,0) \quad \text{or} \quad (2,1).$$

$$\frac{dR}{dt} = 2R\left(1-\frac{R}{4}\right)$$



$$\parallel$$

$$f(R) = 2R - \frac{R^2}{2}$$

$$f'(R) = 2 - R \qquad f'(4) = 2-4 = -2 < 0$$

decoupled system

$$\begin{cases} \frac{dx}{dt} = f(x) \\ \frac{dy}{dt} = g(y) \end{cases} \quad \checkmark$$

partially decoupled system

$$\begin{cases} \frac{dx}{dt} = f(x) \Rightarrow x = \int f(t)\,dt = x(t) \\ \frac{dy}{dt} = g(x,y) \Rightarrow \frac{dy}{dt} = g(x(t), y) = h(t,y) \end{cases}$$

Ex:

coupled system

$$\boxed{\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}} \Leftrightarrow \frac{d}{dt}\frac{dx}{dt} = \frac{dy}{dt} = -x \Leftrightarrow \boxed{\begin{cases} x'' + x = 0 \\ y = x' \end{cases}}$$

$$\frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2} = -x$$

$\boxed{x'' + x = 0}$: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm\sqrt{-1} = \pm i$

un-damped harmonic oscillation

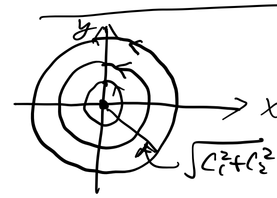
$$\Rightarrow x_1(t) = \cos(t), \quad x_2(t) = \sin(t)$$

$$= e^{0t} \cdot \cos(1 \cdot t)$$

$$\Rightarrow \boxed{x(t) = C_1 \cdot \cos(t) + C_2 \cdot \sin(t)}$$

$$y = x' = -C_1 \sin t + C_2 \cos t$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 \cos t + C_2 \sin t \\ -C_1 \sin t + C_2 \cos t \end{pmatrix} = C_1 \boxed{\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}} + C_2 \boxed{\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}.$$



$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x + y \end{cases} \qquad \frac{d^2x}{dt^2} = \frac{dy}{dt} = x + y = x + \frac{dx}{dt}$$

$$\Updownarrow$$

$$\begin{cases} x'' - x' - x = 0 \\ x' = y \end{cases} \qquad \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2} = \lambda_1, \lambda_2$$

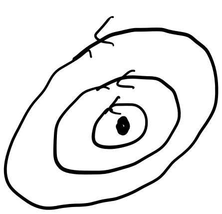
$$\Rightarrow x_1(t) = e^{\lambda_1 t}, \; x_2(t) = e^{\lambda_2 t}$$

$$\Rightarrow x(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t}$$

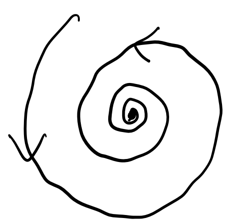
$$\Rightarrow y(t) = x' = c_1 \cdot \lambda_1 \cdot e^{\lambda_1 t} + c_2 \cdot \lambda_2 \cdot e^{\lambda_2 t}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

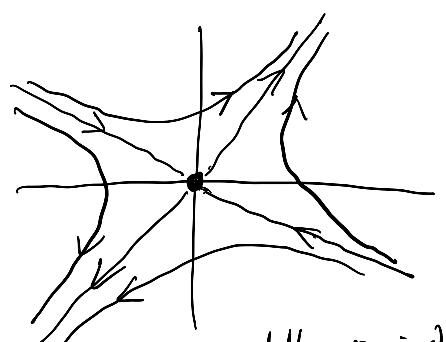
$$\lambda_1 = \frac{1 - \sqrt{5}}{2} < 0 < \lambda_2 = \frac{1 + \sqrt{5}}{2}$$



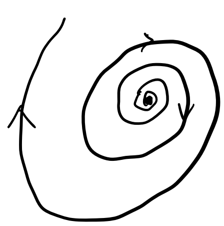
center



sink



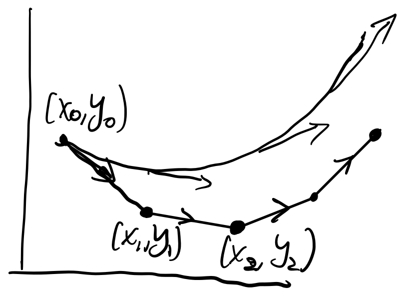
saddle point



source

Euler's method

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$ autonomous system



$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} \Delta t$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} f(x_1, y_1) \\ g(x_1, y_1) \end{pmatrix} \Delta t$$

- - - .

$$\begin{cases} x' = f(x, y, t) \\ y' = g(x, y, t) \end{cases}$$ (t_0, x_0, y_0) non-autonomous system.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} f(x_0, y_0, t_0) \\ g(x_0, y_0, t_0) \end{pmatrix} \Delta t \qquad t_1 = t_0 + \Delta t$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} f(x_1, y_1, t_1) \\ g(x_1, y_1, t_1) \end{pmatrix} \Delta t \qquad t_2 = t_1 + \Delta t$$

- - - .

Existence and Uniqueness theorem.

Thm: $\frac{d\vec{Y}}{dt} = \vec{F}(t, Y) = \begin{pmatrix} f(t, x, y) \\ g(t, x, y) \end{pmatrix}$ $\quad \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$

suppose $\vec{F}$ is continuously differentiable near $(t_0, \vec{Y}_0)$

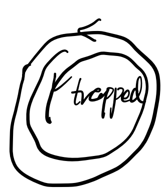
Then there is a unique solution $\vec{Y}(t)$ defined near t_0

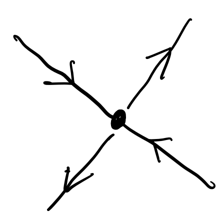
s.t. $\vec{Y}(t_0) = \vec{Y}_0$.

For autonomous system, the vector field does not vary with time.

- the solution curve cannot intersect itself unless the solution is periodic
- two solution curves cannot intersect.

Ex:





5 solution curves divide the phase plane into 4 regions.