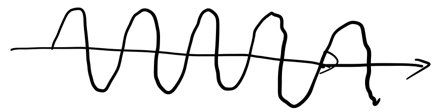
| | undamped | damped |
|---|---|---|
| unforced | $my'' + ky = 0$ | $my'' + by' + ky = 0$ |
| forced | $my'' + ky = f(t)$ | $my'' + by' + ky = f(t)$. |

- undamped + unforced:



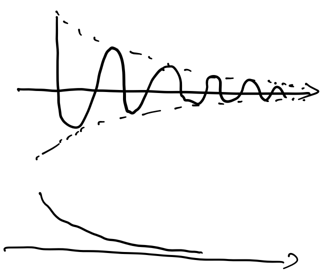
$c_1 \cdot \cos(\omega_0 t) + c_2 \cdot \sin(\omega_0 t)$, $\quad \omega_0 = \sqrt{\frac{k}{m}}$ natural frequency

- damped + unforced:

$b < 2\sqrt{mk}$ underdamped

$b = 2\sqrt{mk}$ critical

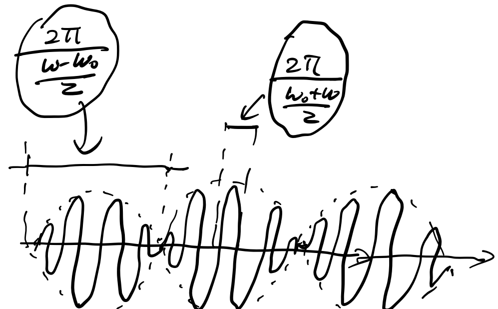
$b > 2\sqrt{mk}$ overdamped.



- undamped + sinusoidal forcing

$$\boxed{my'' + ky = A\cos(\omega t)}$$

, $\omega$ close to $\omega_0$ but $\omega \neq \omega_0$




$y(t) = c_1 \cdot \cos(\omega_0 t) + c_2 \cdot \sin(\omega_0 t) + \boxed{a \cos(\omega t)}$ $y_p$ particular sol. to

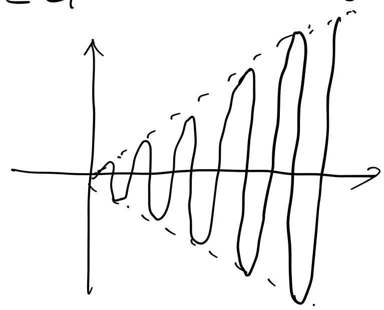
special case: $\boxed{y(0) = y'(0) = 0} \Rightarrow c_1 + a = 0$, $c_2 = 0$ $\Rightarrow c_1 = -a$

$\Rightarrow y(t) = -a\cos(\omega_0 t) + a\cos(\omega t)$

$= \pm a \cdot 2 \cdot \sin\left(\frac{\omega_0 + \omega}{2}\right) t \cdot \sin\left(\frac{\omega - \omega_0}{2}\right) t$.

$-ma\cdot\omega^2 + k\cdot a = A \Rightarrow \quad a = \frac{A}{k - m\omega^2} = \frac{A}{m}\cdot\frac{1}{\omega_0^2 - \omega^2}$

- $\omega = \omega_0 \quad y(t) = C_1\cdot\cos(\omega_0 t) + C_2\sin(\omega_0 t) + t\,(a\cos(\omega t) + b\sin(\omega t))$



- damped + forced oscillation

$m y'' + b y' + k y = A\cdot\cos(\omega t).$

$y(t) = C_1 e^{-\alpha t}\cos(\beta t) + C_2\cdot e^{-\alpha t}\sin(\beta t)$
$+ (a\cdot\cos(\omega t) + b\sin(\omega t))$

$\lambda = -\alpha \pm i\beta$ are the roots to $m\lambda^2 + b\lambda + k = 0$

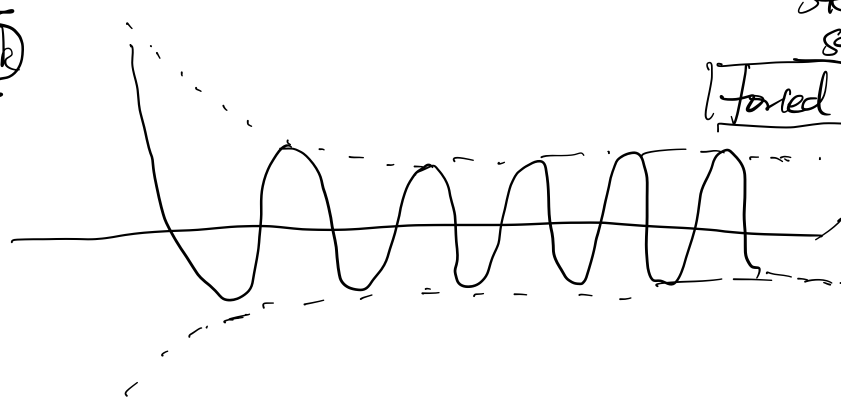
|| assume underdamped case $\Rightarrow \alpha > 0 \quad \beta \neq 0.$

$-\alpha < 0$

$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

steady state

forced response



System of differential equations

$$\boxed{my'' + by' + ky = f(t)} \qquad \text{Set } \underline{v = y'}$$

$$\Longleftrightarrow \begin{cases} y' = v \\ v' = y'' = -\frac{k}{m}y - \frac{b}{m}\underset{v}{\underset{\shortparallel}{y'}} + \frac{f(t)}{m} \end{cases}$$

$$\Longleftrightarrow \begin{cases} y' = v \\ v' = -\frac{k}{m}y - \frac{b}{m}v + \frac{f(t)}{m} \end{cases}$$

$\frac{f(t)}{m} = F(t)$.

$$\Longleftrightarrow \frac{d}{dt}\boxed{\begin{pmatrix} y \\ v \end{pmatrix}} = \underline{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}}\begin{pmatrix} y \\ v \end{pmatrix} + \underline{\begin{pmatrix} 0 \\ \frac{f(t)}{m} \end{pmatrix}}$$

$$\boxed{\frac{d}{dt}Y = A \cdot Y + F(t) \qquad Y = \begin{pmatrix} y \\ v \end{pmatrix}}$$

1st order linear system, nonhomogeneous.

$$\boxed{y''' - 2y' + y = \cos(t).}$$

$\underline{y' = v}$, $y'' = w$

$$\begin{cases} y' = v \\ v' = y'' = w \\ w' = y''' = 2y' - y + \cos(t) \\ \quad = 2v - y + \cos(t). \end{cases}$$

$$\Leftrightarrow \frac{d}{dt}\begin{pmatrix} y \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ -y+2v+\cos(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} y \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cos(t) \end{pmatrix}$$

# Predator – Prey system

F: population of Predator (Fox)

R: - - - - - Prey (Rabbit).  $Y = \begin{pmatrix} R \\ F \end{pmatrix}$

$$\begin{cases} \dfrac{dR}{dt} = k_1 \cdot R - \alpha \cdot FR \\ \dfrac{dF}{dt} = -k_2 F + \beta \cdot F \cdot R \end{cases} \iff \frac{d}{dt} Y = G(Y)$$

$$G\begin{pmatrix} R \\ F \end{pmatrix} = \begin{pmatrix} k_1 R - \alpha FR \\ -k_2 F + \beta FR \end{pmatrix}$$

autonomous system: right-hand-side depends only on Y

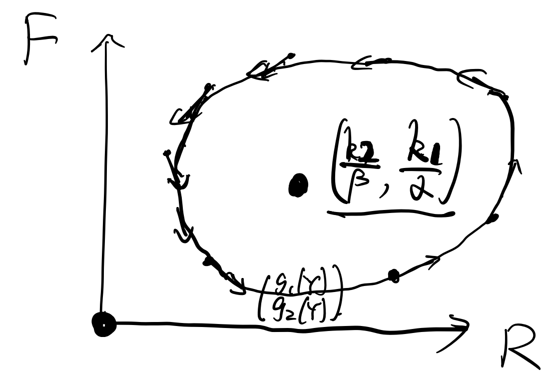
- equilibrium solution: $Y \equiv Y_0 \Rightarrow \frac{dY}{dt} = 0 = G(Y_0)$
  constant solution  ($Y_0$ ← constant)

$$\begin{cases} k_1 R - \alpha \cdot FR = 0 = R \cdot (k_1 - \alpha \cdot F) \Rightarrow R = 0, \text{ or } F = \frac{k_1}{\alpha} \\ -k_2 F + \beta \cdot F \cdot R = 0 = F(-k_2 + \beta \cdot R) \Rightarrow F = 0 \text{ or } R = \frac{k_2}{\beta}. \end{cases}$$

$(R = 0 \text{ and } F = 0)$ or $\left(R = \frac{k_2}{\beta} \text{ and } F = \frac{k_1}{\alpha}\right)$

$\begin{pmatrix} R \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\boxed{\begin{pmatrix} R \\ F \end{pmatrix} = \begin{pmatrix} \frac{k_2}{\beta} \\ \frac{k_1}{\alpha} \end{pmatrix}}$  2 equilibrium solutions.

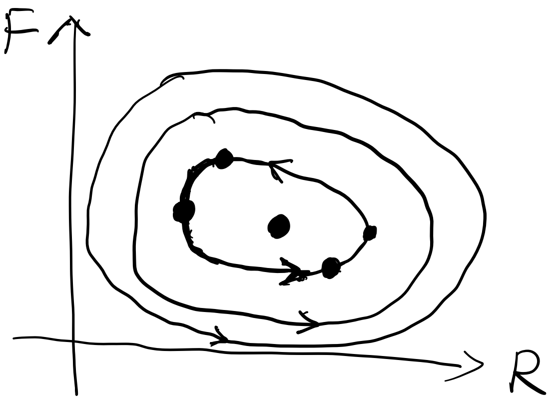
- phase plane




$$\frac{dY}{dt} = G(Y) = \begin{pmatrix} g_1(Y) \\ g_2(Y) \end{pmatrix}$$

→ vector field on the phase plane

$$t \mapsto Y = Y(t) = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix} \qquad \frac{dY}{dt} = G(Y)$$




$$\begin{cases} \dfrac{dR}{dt} = 2R - FR \\ \dfrac{dF}{dt} = -F + \frac{1}{2}FR \end{cases}$$

$$\frac{dF}{dR} = \frac{\frac{dF}{dt}}{\frac{dR}{dt}} = \frac{-F + \frac{1}{2}FR}{2R - FR} = \frac{\frac{1}{2} \cdot F(-2+R)}{R(2-F)}$$

$$\frac{2-F}{F} dF = \frac{1}{2} \cdot \frac{1}{R}(-2+R)\, dR$$

$$\left(\frac{2}{F} - 1\right) dF \qquad \frac{1}{2}\left(-\frac{2}{R} + 1\right) dR$$

$$2 \cdot \ln|F| - F = \frac{1}{2} \cdot (-2 \ln R + R) + C$$

$$\boxed{2 \ln F + \ln R - F - \frac{1}{2} R = C}$$

$F = F(R)$

equilibrium: $(0,0)$, $(2,2)$ (implicit function)

$$F^2 \cdot R \cdot e^{-F - \frac{1}{2} R} = \text{constant}$$