Forced Oscillation.

Ex: $\boxed{y'' + 4y = A\cdot\cos(\omega\cdot t)}$

- $y''+4y=0 \qquad \lambda^2+4=0 \Rightarrow \lambda^2=-4,\ \lambda=\sqrt{-4}=\pm 2i$

$\Rightarrow y_1(t) = \cos(2t) \quad y_2(t)=\sin(2t).$ 2: internal frequency (natural frequency)

internal freq.

- Find a particular solution

Complexify: $y''+4y = A\cdot e^{i\omega t} = A\cdot(\cos(\omega t)+i\cdot\sin(\omega t))$

Case 1: $\omega\neq\pm2$. $y_c(t) = a\cdot e^{i\omega t}, \quad y_c''(t) = (i\omega)^2 a\cdot e^{i\omega t} = -a\omega^2 e^{i\omega t}$

$$y_c''+4y_c = -a\cdot\omega^2 e^{i\omega t} + 4a\cdot e^{i\omega t} = a(4-\omega^2)\cancel{e^{i\omega t}} = A\cdot\cancel{e^{i\omega t}}$$

$\Rightarrow a = \frac{A}{4-\omega^2}$ if $\omega^2\neq4$ i.e. $\omega\neq\pm2$.

$\Rightarrow y_c(t) = \frac{A}{4-\omega^2}\cdot e^{i\omega t} = \frac{A}{4-\omega^2}(\cos(\omega t)+i\sin(\omega t))$

$\Rightarrow y_p(t) = \frac{A}{4-\omega^2}\cos(\omega t)$

$$\boxed{\bullet\ y(t) = y_h + y_p = C_1\cdot\cos(2t) + C_2\cdot\sin(2t) + \frac{A}{4-\omega^2}\cos(\omega t). \quad \text{if } \omega\neq\pm2}$$

Assume $y(0)=y'(0)=0$

$\Downarrow$

$C_1 = -\frac{A}{4-\omega^2},\ C_2=0$

$$\boxed{\underbrace{-k}_{C_1}\cdot\cos(2t) + \underbrace{k}_{\frac{A}{4-\omega^2}}\cdot\cos(\omega t)} = k\cdot(\cos(\omega t) - \cos(2t))$$

$$-\cos(2t)+\cos(\omega t) \rightsquigarrow -e^{i2t}+e^{i\omega t} = -e^{i(\alpha+\beta)t}+e^{i(\alpha-\beta)t}$$

$$\alpha=\frac{2+\omega}{2},\quad \beta=\frac{2-\omega}{2}$$

$$= -e^{i\alpha t}\cdot\left(\frac{e^{i\beta t}-e^{-i\beta t}}{2i}\right)\cdot 2i$$

$$= -e^{i\alpha t}\cdot 2i\cdot\sin(\beta t)$$

$$= -(\cos(\alpha t)+i\sin(\alpha t))\cdot\sin(\beta t)\cdot 2i$$

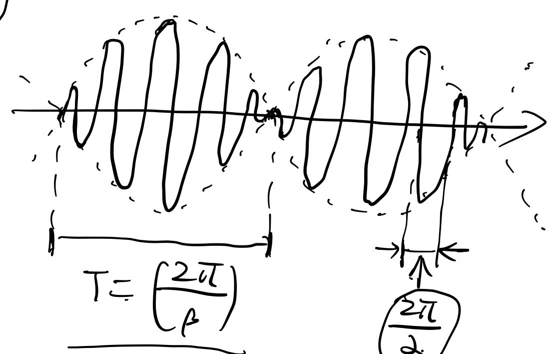
$$= \sin(\alpha t)\cdot\sin(\beta t)2 - i(\dots)$$

beats

$$\boxed{\cos(\omega t)-\cos(2t) = +2\cdot\sin(\alpha t)\cdot\sin(\beta t)}$$

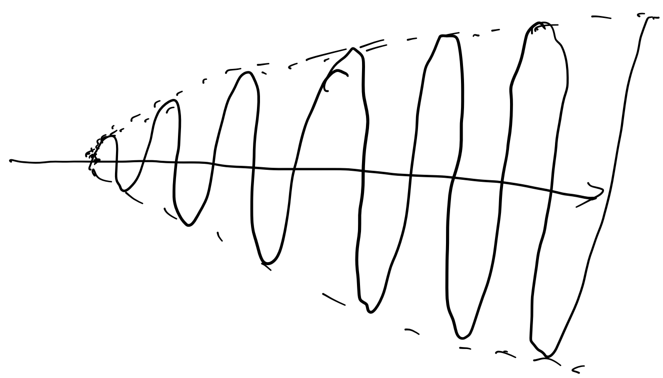
$$\frac{2+\omega}{2} > \frac{2-\omega}{2}$$

smaller period — bigger period




$$\frac{2-\omega}{2}=\beta \xrightarrow{\omega\to 2} 0$$

$$T \to +\infty$$



Case 2: $\omega = 2$. $\quad y'' + 4y = A\cdot e^{i\omega t}$

$$y_c = a\cdot t\cdot e^{2it} \Rightarrow y_c' = a\left(e^{2it} + t\cdot 2i\cdot e^{2it}\right)$$

$$y_c'' = a\cdot\left(4i\cdot e^{2it} + t\cdot(2i)^2 e^{2it}\right)$$

$$= 4a\cdot i\, e^{2it} - 4at\cdot e^{2it}$$

$$y_c'' + 4y_c = 4a\cdot i\, e^{2it} - \cancel{4at}\, e^{2it} + 4\cdot \cancel{at}\, e^{2it} = A\cdot e^{2it}$$

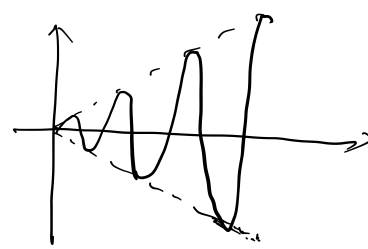
$$\Rightarrow 4ai = A \Rightarrow a = \frac{A}{4i} = -\frac{A}{4} i.$$

$$\Rightarrow y_c = -\frac{A}{4} i\cdot t\cdot e^{2it} = -\frac{A}{4} i\cdot t\cdot\left(\cos(2t) + i\cdot\sin(2t)\right)$$

$$= \frac{A}{4} t\cdot\sin(2t) - i\cdot\frac{A}{4} t\cdot\cos(2t))$$

$$\Rightarrow y_p = \frac{A}{4} t\cdot\sin(2t)$$

$$\Rightarrow y = y_h + y_p = c_1\cdot\cos(2t) + c_2\cdot\sin(2t) + \frac{A}{4} t\, \sin(2t).$$



resonance

$\omega = 2$

Damped oscillation:

$$y'' + by' + 4y = A\cdot\cos(\omega t). \qquad \boxed{b>0}$$

- $y'' + by' + 4y = 0.$

$$\lambda^2 + b\lambda + 4 = 0 \Rightarrow \lambda = \frac{-b \pm \sqrt{b^2-16}}{2}$$

underdamped case: $b^2 < 16 \Rightarrow \lambda = -\frac{b}{2} \pm i\cdot\frac{\sqrt{16-b^2}}{2}$

$$= -u \pm iv \qquad \boxed{u>0}$$

$$\Rightarrow \boxed{\begin{array}{l} y_1(t) = e^{-ut}\cdot\cos(vt) \\ \hline y_2(t) = e^{-ut}\cdot\sin(vt). \end{array}}$$

- $y'' + by' + 4y = A\cdot e^{i\omega t}$

$y_c(t) = a\cdot e^{i\omega t}$, $\quad y_c' = a\cdot i\omega\cdot e^{i\omega t}$

$$y_c'' = -a\omega^2 e^{i\omega t}$$

$$-a\omega^2 e^{i\omega t} + b a\cdot i\omega\cdot e^{i\omega t} + 4\cdot a\cdot e^{i\omega t} = A\cdot e^{i\omega t}$$

$$e^{i\omega t}\cdot a\cdot(-\omega^2 + ib\omega + 4) = A\cdot e^{i\omega t}$$

$$\Rightarrow a = \frac{A}{4-\omega^2+ib\omega}$$

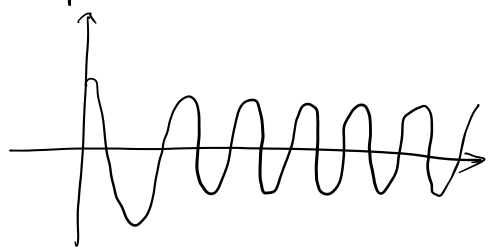
$$\Rightarrow \; y_c(t) = \frac{A}{4-\omega^2+ib\omega}\cdot e^{i\omega t} = \underbrace{y_p}_{\boxed{f\cdot\cos(\omega t)+g\cdot\sin(\omega t)}} + i\cdot y_q .$$

$$y = C_1\cdot y_1 + C_2\cdot y_2 + y_p$$

$$= \underbrace{C_1\cdot e^{-ut}\cos(vt) + C_2\cdot e^{-ut}\sin(vt)}_{\text{natural (internal) response}} + \underbrace{\boxed{f\cos(\omega t)+g\cdot\sin(\omega t)}}_{\text{forced response } = \text{ steady state}}$$

$u>0$: $C_1\cdot e^{-ut}\cos(vt) + C_2\cdot e^{-ut}\sin(vt) \xrightarrow{t\to+\infty} 0$

$f\cos(\omega t)+g\cdot\sin(\omega t) = A\cdot\cos(\omega t+\phi)$.



$$f\cos(\omega t)+g\cdot\sin(\omega t) = \Big(\underbrace{\frac{f}{\sqrt{f^2+g^2}}}_{\cos(\theta)}\cos(\omega t) + \underbrace{\frac{g}{\sqrt{f^2+g^2}}}_{\sin(\theta)}\sin(\omega t)\Big)\cdot\underbrace{\sqrt{f^2+g^2}}_{A}$$

$$= A\cdot\cos(\omega t-\theta) = A\cdot\cos(\omega t+\phi)$$

$\phi$: phase angle

$$y'' + 2y' + 4y = \cos(2t).$$

$y' = v$

$v' = y''$

$= -2y' - 4y + \cos(2t)$

$= -2v - 4y + \cos(2t)$

$$\Leftrightarrow \begin{cases} y' = v \\ v' = -4y - 2v + \cos(2t). \end{cases}$$

$$\Leftrightarrow \frac{d}{dt}\underbrace{\begin{pmatrix} y \\ v \end{pmatrix}}_{= Y} = \underbrace{\begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix}}_{= A} \begin{pmatrix} y \\ v \end{pmatrix} + \underline{\begin{pmatrix} 0 \\ \cos(2t) \end{pmatrix}}.$$

$$\Leftrightarrow \boxed{\frac{d}{dt} Y = A \cdot Y + F(t).} \qquad Y = \begin{pmatrix} y \\ v \end{pmatrix}.$$

1st order linear system, nonhomogeneous.

$\rightsquigarrow$

- $\frac{dY}{dt} = AY \rightsquigarrow Y_1(t), Y_2(t).$
- $Y_p$ particular to nonhom. eq.
- $Y = C_1 Y_1 + C_2 Y_2 + Y_p.$