Example: $y'' + 5y' + 4y = e^{-t}$, $y(0)=1$, $y'(0)=-1$.

- associated homogeneous equation $y''+5y'+4y=0$

  characteristic polynomial: $\lambda^2+5\lambda+4=0$.

  $\Rightarrow$ roots: $\lambda_1=-1$, $\lambda_2=-4$

  $\Rightarrow$ basic solutions: $y_1(t)=e^{-t}$, $y_2(t)=e^{-4t}$

  $\Rightarrow$ $y_h = C_1 \cdot e^{-t} + C_2 e^{-4t}$ is the general solution to the hom. eq.

- Find a particular solution: -1 is a root $\Rightarrow$ $a \cdot e^{-t}$ does not work.

  Need to use $y_p(t) = a \cdot t \cdot e^{-t}$ $\Rightarrow$ $y_p' = a \cdot (e^{-t} - t \cdot e^{-t})$

  $$y_p'' = a \cdot (-e^{-t} - e^{-t} + t \cdot e^{-t})$$

  $$\Rightarrow \quad y_p'' + 5y_p' + 4y_p = a \cdot (-2e^{-t} + \cancel{te^{-t}}) + 5a(e^{-t} - \cancel{te^{-t}}) + 4a\cancel{te^{-t}}$$

  $$= 3a \cdot e^{-t} = e^{-t} \Rightarrow 3a = 1 \Rightarrow a = \frac{1}{3}$$

  $\Rightarrow$ $y_p(t) = \frac{1}{3} t \cdot e^{-t}$ is a particular solution to the non-hom. eq.

- General solution to the non-homogeneous equation:

  $$y(t) = y_h(t) + y_p(t) = C_1 e^{-t} + C_2 \cdot e^{-4t} + \frac{1}{3} t \cdot e^{-t}$$

  Use initial condition: $\begin{cases} 1 = y(0) = C_1 + C_2 \\ -1 = y'(0) = -C_1 - 4C_2 + \frac{1}{3} \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = 1 \\ C_1 + 4C_2 = \frac{4}{3} \end{cases} \Rightarrow \begin{cases} C_1 = \frac{8}{9} \\ C_2 = \frac{1}{9} \end{cases}$

  $\Rightarrow$ unique solution: $y(t) = \frac{8}{9} e^{-t} + \frac{1}{9} e^{-4t} + \frac{1}{3} t \cdot e^{-t}$.

<u>Example:</u> $y'' + 5y' + 4y = \cos(2t)$. $y(0)=1$, $y'(0)=-1$.

- general solution to the homogeneous equation:

$$y(t) = C_1 \cdot e^{-t} + C_2 \cdot e^{-4t}$$

- complexification : $y'' + 5y' + 4y = e^{2it} = \underbrace{\cos(2t)}_{\text{real part}} + i \cdot \sin(2t)$.

$2i \neq -1$ or -4 $\Rightarrow$ use $\tilde{y}_p(t) = a \cdot e^{2it}$

$$\Rightarrow \tilde{y}_p' = a \cdot 2i \cdot e^{2it}, \quad \tilde{y}_p'' = a \cdot (2i)^2 e^{2it} = -4a \cdot e^{2it}$$

$$\Rightarrow \underset{\overset{\|}{e^{2it}}}{\tilde{y}_p'' + 5\tilde{y}_p' + 4\tilde{y}_p} = -4a \cdot e^{2it} + 10 a i \cdot e^{2it} + 4a \cdot e^{2it}$$

$$= 10 a \cdot i \cdot e^{2it}$$

$$\Rightarrow 10 a i = 1 \Rightarrow a = \frac{1}{10 i} = -\frac{1}{10} i$$

$$\Rightarrow \tilde{y}_p(t) = -\frac{1}{10} i\, e^{2it} = -\frac{1}{10} i \cdot (\cos(2t) + i \sin(2t))$$

$$= +\frac{1}{10} \sin(2t) - \frac{i}{10} \cos(2t)$$

$\Rightarrow$ real part of $\tilde{y}_p(t)$: $\frac{1}{10}\sin(2t)$ is a particular solution to the real non-hom eq.

- general solution $y(t) = y_h(t) + y_p(t) = C_1 e^{-t} + C_2 \cdot e^{-4t} + \frac{1}{10}\sin(2t)$.

use initial condition : $1 = y(0) = C_1 + C_2$

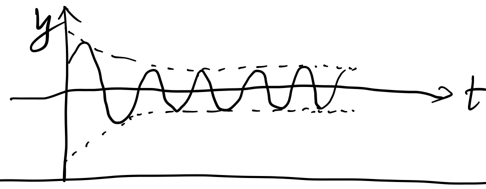
$$-1 = y'(0) = -C_1 - 4C_2 + \frac{1}{5} \Rightarrow \begin{cases} C_1 + C_2 = 1 \\ C_1 + 4C_2 = \frac{6}{5} \end{cases}$$

$$\Rightarrow \begin{cases} C_1 = \frac{14}{15} \\ C_2 = \frac{1}{15} \end{cases}$$

$\Rightarrow$ unique solution

$$y(t) = \overbrace{\frac{14}{15} e^{-t} + \frac{1}{15} e^{-4t}}^{\text{as } t \to +\infty \text{ converges to } 0} + \underline{\frac{1}{10}\sin(2t)}.$$

$\leftarrow$ steady state.

Graph: 

Ex: $y'' + 4y = \cos(2t)$. $\quad y(0) = 0 = y'(0)$.

- $y'' + 4y = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

  $\Rightarrow y_1(t) = \cos(2t),\ y_2(t) = \sin(2t)$

  $\Rightarrow y(t) = C_1 \cdot \cos(2t) + C_2 \cdot \sin(2t)$.

- complexification: $y'' + 4y = e^{2it} = \cos(2t) + i\sin(2t)$.

  $2i$ is a root (of multiplicity 1) $\Rightarrow$ need use $\tilde{y}_p(t) = a \cdot t \cdot e^{2it}$

  $\Rightarrow \tilde{y}_p' = a(e^{2it} + t \cdot 2i e^{2it})$, $\tilde{y}_p'' = a((2i)e^{2it} + 2i e^{2it} - 4te^{2it})$

  $\Rightarrow \tilde{y}_p'' + 4\tilde{y}_p = a(4i e^{2it} - \cancel{4te^{2it}}) + \cancel{4a \cdot t e^{2it}}$

  $\overset{\parallel}{e^{2it}} \quad = a \cdot 4i \cdot e^{2it}$

  $\Rightarrow a \cdot 4i = 1 \Rightarrow a = \frac{1}{4i} = -\frac{1}{4}i$

  $\Rightarrow \tilde{y}_p(t) = -\frac{1}{4} i t \cdot e^{2it} = -\frac{1}{4} i \cdot t(\cos(2t) + i\sin(2t))$

  $= \frac{1}{4} t \sin(2t) - \frac{i}{4} t\cos(2t)$

  $\Rightarrow y_p(t) = \frac{1}{4} t \cdot \sin(2t)$ is a particular solution

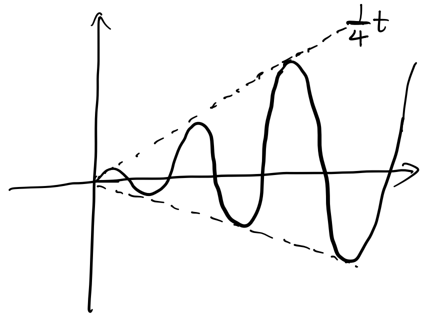
- $y(t) = y_h + y_p = C_1 \cdot \cos(2t) + C_2 \cdot \sin(2t) + \frac{1}{4} t \sin(2t)$.

  $0 = y(0) = C_1 \Rightarrow C_1 = 0$

  $0 = y'(0) = C_2 \cdot 2 \Rightarrow C_2 = 0$

  $\Rightarrow y(t) = \frac{1}{4} t \sin(2t)$ is the unique solution.

Resonance:



In general: $y'' + \left(\frac{k}{m}\right) y = \cos(\omega t)$

$\frac{k}{m} = \omega_0^2$

$\omega_0 = \sqrt{\frac{k}{m}}$
internal frequency
(natural)

Resonance happens when $\omega = \omega_0$

General rule to guess the particular solution:

$$y'' + by' + c\,y = e^{kt}\cos(\omega t)\cdot P_d(t)$$

$P_d(t)$: polynomial of degree d.

- $\lambda^2 + b\lambda + c = 0 \Rightarrow \lambda = \dfrac{-b \pm \sqrt{b^2-4c}}{2}$, $\lambda = \lambda_1, \lambda_2$

case 1: $b^2 - 4c > 0$. $\lambda_1 \neq \lambda_2$ real $\Rightarrow y_1(t) = e^{\lambda_1 t}$, $y_2(t) = e^{\lambda_2 t}$

case 2: $b^2 - 4c = 0$. $\lambda_1 = \lambda_2$ real root of multiplicity 2

$\Rightarrow y_1(t) = e^{\lambda_1 t}$, $y_2(t) = t\cdot e^{\lambda_1 t}$

case 3: $b^2 - 4c < 0$. $\lambda = \alpha \pm i\beta$ with $\beta \neq 0$

$\Rightarrow y_1(t) = e^{\alpha t}\cos(\beta t)$, $y_2(t) = e^{\alpha t}\sin(\beta t)$.

- complexification: $y'' + by' + cy = e^{kt} \cdot e^{i\omega t} P_d(t)$
$$= e^{(k+i\omega)t} P_d(t).$$

<u>case 1</u>: $k+i\omega$ is not a root to characteristic polynomial

$$\Rightarrow \tilde{y}_p(t) = e^{(k+i\omega)t} \underbrace{Q_d(t)}$$
undetermined polynomial of degree d

<u>case 2</u>: $k+i\omega$ is a root of multiplicity 1

$$\Rightarrow \tilde{y}_p(t) = t \cdot e^{(k+i\omega)t} Q_d(t)$$

<u>case 3</u>: $k+i\omega = \underset{\lambda_1 = \lambda_2}{\underset{\shortparallel}{k}}$ is a root of multiplicity 2

$$\Rightarrow \tilde{y}_p(t) = t^2 \cdot e^{(k+i\omega)t} Q_d(t)$$

<u>Ex</u>: $y'' + 4y' + 4y = e^{-2t}$

- $\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda = -2$ multiplicity 2
$$\Rightarrow y_1(t) = e^{-2t},\ y_2(t) = t \cdot e^{-2t}$$

- -2 is a root of multiplicity 2

$\Rightarrow$ need to use $y_p(t) = a \cdot t^2 \cdot e^{-2t}$.