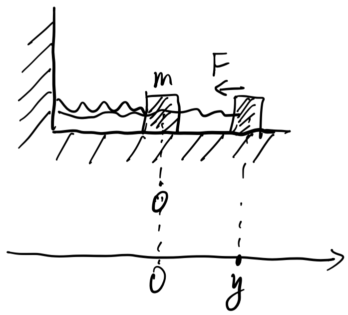


$F = -k \cdot y$ Hooke's law

$F = m \cdot a$ Newton's law

$a = \frac{d^2y}{dt^2} = \frac{dv}{dt}$ $v = \frac{dy}{dt}$

$$m \cdot \frac{d^2y}{dt^2} = -k \cdot y \iff m\frac{d^2y}{dt^2} + k \cdot y = 0 \iff \underbrace{\frac{d^2y}{dt^2} + \overset{\omega^2}{\overset{\parallel}{\frac{k}{m}}} y = 0}_{\text{homogeneous}}.$$

Solve: $y'' + \omega^2 y = 0$. $\omega = \sqrt{\frac{k}{m}}$

- $\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \sqrt{-\omega^2} = \pm \omega i$ $(i = \sqrt{-1},\ i^2 = -1)$

  characteristic polynomial

- basic solution: $\tilde{y}_1 = e^{i\omega t} = \cos(\omega t) + i \cdot \sin(\omega t)$.

  $\tilde{y}_2 = e^{-i\omega t} = \cos(\omega t) - i \cdot \sin(\omega t)$.

$y_1 = \frac{\tilde{y}_1 + \tilde{y}_2}{2} = \cos(\omega t)$ basic solutions to the homogeneous eq.

$y_2 = \frac{\tilde{y}_1 - \tilde{y}_2}{2i} = \sin(\omega t)$

- general solution $y(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t)$

  $= C_1 \cos(\omega t) + C_2 \cdot \sin(\omega t)$ $C_1, C_2 \in \mathbb{R}$.

- Initial condition $\left| y(0) = y_0,\ y'(0) = v_0 \right.$

$y_0 = C_1 \cdot 1 + C_2 \cdot 0 = C_1$, $y'(t) = -C_1 \cdot \omega \cdot \sin(\omega t) + C_2 \cdot \omega \cdot \cos(\omega t)$.

$v_0 = -C_1 \cdot 0 + C_2 \cdot \omega = C_2 \cdot \omega \Rightarrow C_2 = \frac{v_0}{\omega}$.

$$\Rightarrow \quad y(t) = \underbrace{y_0}_{a}\cdot\cos(\omega t) + \underbrace{\frac{v_0}{\omega}}_{b}\cdot\sin(\omega t).$$

$$= a\cdot\cos(\omega t) + b\cdot\sin(\omega t).$$

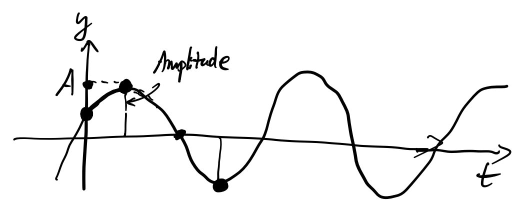
$$= \left(\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\cos(\theta)}\cdot\cos(\omega t) + \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\sin(\theta)}\sin(\omega t)\right)\cdot\sqrt{a^2+b^2}$$

$$\tan(\theta) = \frac{b}{a}$$

$$= \left(\sqrt{a^2+b^2}\right)\cdot\left(\cos(\omega t)\cdot\cos(\theta) + \sin(\omega t)\cdot\sin(\theta)\right)$$

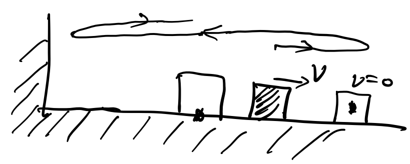
$$= \sqrt{a^2+b^2}\cdot\left(\underbrace{\cos(\omega t-\theta)}_{\cos\left(\omega\left(t-\frac{\theta}{\omega}\right)\right)}\right)$$

$\sqrt{a^2+b^2}$ = Amplitude $\quad \cos\left(\omega\left(t-\frac{\theta}{\omega}\right)\right)$



period: $\frac{2\pi}{\omega}$, ω: Frequency.

$$\omega = \sqrt{\frac{k}{m}}$$



- $$\underbrace{m\cdot y''}_{\text{mass x acceleration}} = \underbrace{-k\cdot y}_{\text{force of the spring}} - \underbrace{b\cdot y'}_{\text{damping force}}$$

<u>Solve</u>: $m y'' + b\cdot y' + k\cdot y = 0$.

$m>0$

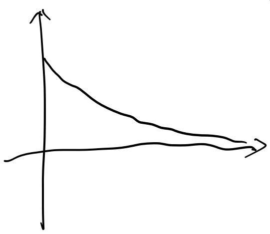
$b>0$

$k>0$

· characteristic polynomial

$$m \cdot \lambda^2 + b\lambda + k = 0 \Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

· Overdamping case: $\boxed{b^2 - 4mk > 0}$, $\boxed{\lambda_1 < \lambda_2 < 0}$



$\Rightarrow y_1(t) = e^{\lambda_1 t}$, $y_2(t) = e^{\lambda_2 t}$

general solution $y(t) = \underline{C_1 \cdot e^{\lambda_1 t} + C_2 \cdot e^{\lambda_2 t}}$

$\Rightarrow \lim_{t \to +\infty} y(t) = 0.$

$0 > \alpha = -\frac{b}{2m}$, $\beta = \frac{\sqrt{4mk - b^2}}{2m}$, $\beta > 0$

$\lambda_1 = \alpha + i\beta$

$\lambda_2 = \alpha - i\beta.$

· underdamping: $\boxed{b^2 - 4mk < 0}$ $(b^2 < 4mk)$

$$\tilde{y}_1(t) = e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos(\beta t) + i \cdot \sin(\beta t)).$$

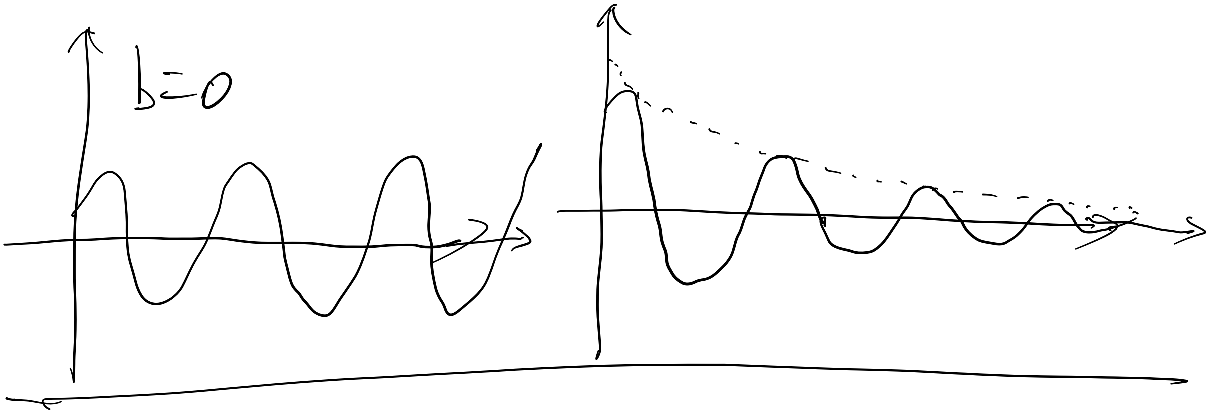
$$\tilde{y}_2(t) = \ldots = e^{\alpha t}(\cos(\beta t) - i \sin(\beta t))$$

$$y_1(t) = e^{\alpha t} \cdot \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).$$

$$\Rightarrow y(t) = C_1 e^{\alpha t} \cdot \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t). \quad \boxed{\alpha = -\frac{b}{2m} < 0.}$$

$$= e^{\alpha t} \cdot (C_1 \cdot \cos(\beta t) + C_2 \cdot \sin(\beta t)).$$

$$= \left(e^{\alpha t}\underline{\sqrt{C_1^2+C_2^2}}\right)\cos(\beta t-\theta) \quad \left(\tan\theta = \frac{C_2}{C_1}\right)$$



• critical case: $\underline{b^2-4mk=0}$.

$$m y'' + b\cdot y' + k y = 0 \qquad \lambda = \frac{-b \pm \sqrt{b^2-4mk}}{2m}$$

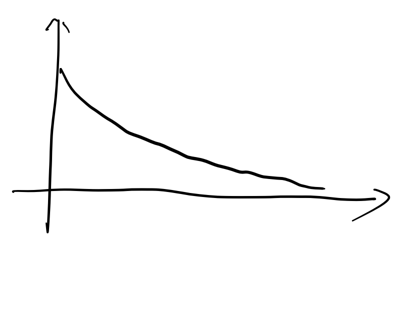
$$\Rightarrow \lambda_1 = \lambda_2 = -\frac{b}{2m} \quad (\text{one root with multiplicity } \underline{2})$$

$$y_1(t) = e^{\lambda_1 t} = e^{-\frac{b}{2m}t}$$

$$y_2(t) = \underline{t}\cdot e^{\lambda_1 t} = t\cdot e^{-\frac{b}{2m}t}$$

$$\Rightarrow y(t) = C_1\cdot y_1(t) + C_2\cdot y_2(t) = C_1\cdot \underline{e^{-\frac{b}{2m}t}} + \underline{C_2\cdot t e^{-\frac{b}{2m}t}}$$

$$\lim_{t\to+\infty} y(t) = 0.$$



$$\boxed{my'' + by' + ky = 0}.$$

$$\Leftrightarrow \quad \underline{y'' + \frac{b}{m}y' + \frac{k}{m}y = 0}. \qquad \boxed{v = y' = v(t)}.$$

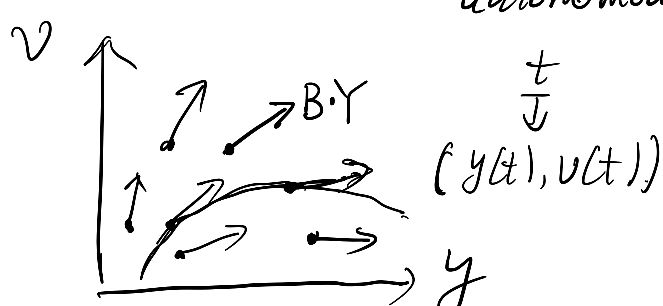
$$\Leftrightarrow \begin{cases} y' = v \\ v' = y'' = -\frac{k}{m}y - \frac{b}{m}y' = -\frac{k}{m}y - \frac{b}{m}\cdot v \end{cases}$$

$$\Leftrightarrow \boxed{\begin{cases} y' = v \\ v' = -\frac{k}{m}y - \frac{b}{m}v. \end{cases}}$$ 1st order (linear) system

$$\Leftrightarrow \frac{d}{dt}\boxed{\begin{pmatrix} y \\ v \end{pmatrix}} = \underline{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}}\begin{pmatrix} y \\ v \end{pmatrix}$$

$$\Leftrightarrow \boxed{\frac{d}{dt}Y = B\cdot Y} \qquad Y = \begin{pmatrix} y \\ v \end{pmatrix}$$

autonomous.



phase plane

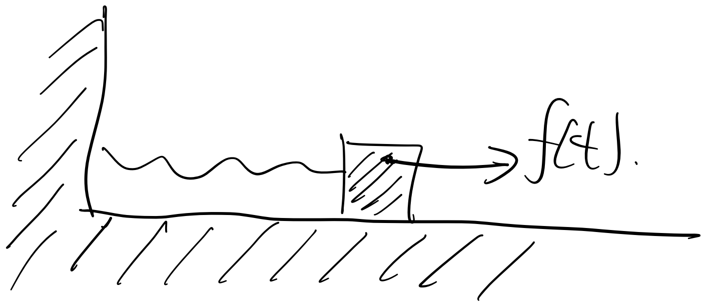
$\frac{dy}{dt} = f(y)$

$$\boxed{1 \cdot y'' + by' + 4y = 0}$$

$b^2 - 4mk = b^2 - 4 \times 1 \times 4 = b^2 - 16$

- $b^2 - 16 > 0$, $b > 4$ over-damping
- $b^2 - 16 < 0$, $b < 4$ under-damping
- $b^2 - 16 = 0$ $b = 4$ critical.

$$\boxed{\begin{cases} y' = v \\ v' = -4y - bv. \end{cases}}$$



$$\underline{m\cdot y''} = -k\cdot y - b\cdot y' + f(t).$$

$$m y'' + b y' + k y = f(t).$$

nonhomogeneous term.

(Superposition) linearity Principle

$$y = y_h + \boxed{y_p} = c_1\cdot \underline{y_1} + c_2\cdot \underline{y_2} + y_p$$

sol. to hom.

basic solutions to hom. eq.

$$\boxed{\underline{Ex}: \quad y'' + 2y' + 4y = e^{-t}}$$

- Find a particular solution y_p.

Guess $y_p = a \cdot e^{-t}$

$$y_p' = -a \cdot e^{-t}, \quad y_p'' = a \cdot e^{-t}$$

$$a e^{-t} + 2 \cdot (-a \cdot e^{-t}) + 4 \cdot a \cdot e^{-t} = e^{-t}$$

$$(a - 2a + 4a) \cdot e^{-t} = 3a \cdot e^{-t}$$

$$\Rightarrow a = \frac{1}{3} \Rightarrow \boxed{y_p = \frac{1}{3} e^{-t}}$$

- hom.eq. $y'' + 2y' + 4y = 0$

$$\lambda^2 + 2\lambda + 4 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4\times 4}}{2}$$

$$\alpha + i\beta$$

$$= -1 \pm \frac{\sqrt{-12}}{2} = -1 \pm \sqrt{3}\, i$$

$$\Rightarrow \quad y_1(t) = e^{-t} \cdot \cos(\sqrt{3}t)$$

$$= e^{\alpha t} \cdot \cos(\beta t).$$

$$y_2(t) = e^{-t} \sin(\sqrt{3}t).$$

$$y_h = C_1 \cdot y_1 + C_2 \cdot y_2$$

$$= C_1 e^{-t} \cos(\sqrt{3}t) + C_2 \cdot e^{-t} \sin(\sqrt{3}t)$$

- General solution to the original non-hom. eq. is

$$\boxed{\begin{aligned} y &= y_h + y_p \\ &= \underline{c_1} \cdot e^{-t}\cos(\sqrt{3}t) + \underline{c_2} \cdot e^{-t}\sin(\sqrt{3}t) \\ &\quad + \frac{1}{3}e^{-t}. \end{aligned}}$$