2nd order linear DE

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = 0 \quad \text{(homogeneous)}$$

- Linearity Principle (Superposition Principle)

  If $y_1(t)$ and $y_2(t)$ are solutions (to the hom. DE equation), then

  $y(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t)$ is a solution.

- Theorem (form of the general solution). $\{y_1, y_2\}$ basic solutions

  If $y_1(t)$ and $y_2(t)$ are solutions, and if they are linearly independent

  Then the general solution to the homogeneous equation is given by

  $$\boxed{y(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t)}$$ where $C_1, C_2$ are (arbitrary) constants

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Def: $y_1(t)$ and $y_2(t)$ are linearly independent if

- $y_1(t) \not\equiv 0$, $y_2(t) \not\equiv 0$
- $y_2(t)/y_1(t) \not\equiv$ a constant (i.e. $y_2(t) \not\equiv k \cdot y_1(t)$ for any $k$)

Ex: $$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} - 7y = 0.$$

Guess: $y(t) = e^{\lambda t}$ $\quad \frac{dy}{dt} = \lambda e^{\lambda t}$, $\frac{d^2y}{dt^2} = \lambda^2 e^{\lambda t}$

$$\lambda^2 \cdot e^{\lambda t} - 6\lambda e^{\lambda t} - 7\lambda e^{\lambda t} = 0 \overset{e^{\lambda t} \neq 0}{\Longrightarrow} \lambda^2 - 6\lambda - 7 = 0 \Rightarrow \lambda = 7 \text{ or } -1$$

$$= e^{\lambda t} \cdot (\lambda^2 - 6\lambda - 7) = 0$$

$$\lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$$

characteristic polynomial (associated to the equation)

$\Rightarrow y_1(t) = e^{7t}, \quad y_2(t) = e^{-t}$ $\qquad y_1/y_2 = e^{8t} \neq$ constant.

linearly independent

$\overset{\text{Thm}}{\Longrightarrow} \quad y(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t) = \underline{C_1 e^{7t} + C_2 \cdot e^{-t}}$.

IVP: $\begin{cases} y(0) = 1, \Rightarrow 1 = C_1 + C_2 \\ y'(0) = 2 \Rightarrow 2 = 7C_1 - C_2 \end{cases} \Rightarrow 8C_1 = 3 \Rightarrow C_1 = \frac{3}{8} \Rightarrow C_2 = \frac{5}{8}$

$C_2 = 1 - C_1$

$y'(t) = 7C_1 e^{7t} - C_2 \cdot e^{-t}$

$\Rightarrow y(t) = \frac{3}{8} e^{7t} + \frac{5}{8} e^{-t}$ unique sol. to $\begin{cases} y'' - 6y' + 7y = 0 \\ y(0) = 1 \\ y'(0) = 2. \end{cases}$

Ex: $\begin{cases} y'' - 4y' + 4y = 0 \\ y(0) = 1, \ y'(0) = 2 \end{cases}$

step 1: $y = e^{\lambda t} \rightsquigarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2.$

$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

$\Rightarrow \underline{e^{2t} = y_1(t)}.$

$y(t) = te^{at}$

$y' = e^{at} + at \cdot e^{at}$

$y'' = \underbrace{a e^{at} + a e^{at}}_{2a \cdot e^{at}} + a^2 t \cdot e^{at}$

$$2a e^{at} + a^2 t e^{at} - 4\cdot(e^{at} + at e^{at}) + 4te^{at} = 0$$

$$\parallel$$

$$(a^2-4a+4)te^{at} + (2a-4)e^{at}$$

$$\Rightarrow \begin{cases} a^2-4a+4=0 \\ 2a-4=0 \end{cases} \Rightarrow a=2. \Rightarrow \frac{y_2(t) = t\cdot e^{2t}}{y_1(t) = e^{2t}}$$

linearly independent

$$\Rightarrow y(t) = C_1 e^{2t} + C_2 t e^{2t}$$

$$y'(t) = 2C_1 e^{2t} + C_2(e^{2t} + t\cdot 2e^{2t})$$

$$1 = y(0) = C_1$$

$$2 = y'(0) = 2C_1 + C_2 \Rightarrow C_2 = 2-2C_1 = 0.$$

$$\Rightarrow y(t) = e^{2t}$$

$$y'' + p\cdot y' + q\cdot y = 0 \rightsquigarrow \lambda^2 + p\lambda + q = 0$$

$$\Rightarrow \lambda = \frac{-p \pm \sqrt{p^2-4q}}{2}$$

Three Cases:

1. $p^2-4q>0$. $\lambda_1 \neq \lambda_2 \in \mathbb{R} \rightarrow \begin{cases} y_1(t)=e^{\lambda_1 t} \\ y_2(t)=e^{\lambda_2 t} \end{cases}$

2. $p^2-4q=0$ $\lambda_1=\lambda_2=-\frac{p}{2} \rightsquigarrow \begin{cases} y_1(t)=e^{\lambda_1 t} \\ y_2(t)=t\cdot e^{\lambda_1 t} \end{cases}$

3. $p^2-4q<0$. $\lambda_1=a+bi$, $\lambda_2=a-bi$ $\rightsquigarrow \begin{cases} y_1(t)=e^{at}\cdot\cos(bt) \\ y_2(t)=e^{at}\cdot\sin(bt). \end{cases}$

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$\sqrt{-12}=\sqrt{4\times(-3)}=2\sqrt{-3}$

Ex: $y''-2y'+4y=0$.

$\rightsquigarrow \lambda^2-2\lambda+4=0$. $\lambda=\frac{2\pm\sqrt{2^2-4\times4}}{2}=\frac{2\pm\sqrt{-12}}{2}$

$(\lambda-1)^2=-3$

$=1\pm\sqrt{-3}=1\pm\sqrt{3}\,i$ (imaginary number)

$\tilde{y}_1(t)=e^{\lambda_1 t}=e^{(1+\sqrt{3}i)t}=e^t\cdot e^{i\cdot\sqrt{3}t}$

(Euler's formula: $e^{i\theta}=\cos\theta+i\cdot\sin\theta$)

$=e^t(\cos(\sqrt{3}t)+i\cdot\sin(\sqrt{3}t))$.

$=e^t\cos(\sqrt{3}t)+i\cdot e^t\sin(\sqrt{3}t)$.

$\tilde{y}_2(t)=e^t\cos(\sqrt{3}t)-ie^t\sin(\sqrt{3}t)$.

$\frac{\tilde{y}_1(t)+\tilde{y}_2(t)}{2}=e^t\cdot\cos(\sqrt{3}t)=y_1(t)$

$\frac{\tilde{y}_1-\tilde{y}_2(t)}{2i}=e^t\sin(\sqrt{3}t)=y_2(t)$.

$\Rightarrow y(t)=C_1\cdot y_1(t)+C_2\,y_2(t)$

$=C_1\cdot e^t\cos(\sqrt{3}t)+C_2\cdot e^t\sin(\sqrt{3}t)$.

$$y'(t) = c_1 \cdot e^t \cdot \cos(\sqrt{3}t) - c_1 \cdot e^t \cdot \sqrt{3}\sin(\sqrt{3}t)$$
$$+ c_2 \cdot e^t \sin(\sqrt{3}t) + c_2 \cdot e^t \sqrt{3} \cdot \cos(\sqrt{3}t).$$

$$1 = y(0) = c_1$$
$$2 = y'(0) = c_1 + c_2 \cdot \sqrt{3} \Rightarrow c_2 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\Rightarrow y(t) = e^t \cos(\sqrt{3}t) + \frac{\sqrt{3}}{3} e^t \sin(\sqrt{3}t)$$

unique solution to $\begin{cases} y'' - 2y' + 4y = 0 \\ y(0) = 1, \ y'(0) = 2. \end{cases}$

$$\left( \begin{array}{ll} y' + p(t) \cdot y = 0 \text{ separable} \Rightarrow y(t) = e^{-\int p(t)dt} \\ y' = -p(t) \cdot y \qquad\qquad = c_1 \, y_1(t). \end{array} \right)$$